Optimal Filtering
Exercise 1

Gustaf Hendeby*
gustaf.hendeby@liu.se

Version: 2023-11-11

Problem 1.1 Consider the following block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $|A| \neq 0$, $|D| \neq 0$, and $|A|$ denotes the determinant of the matrix A.

a) Derive the block UDL factorization of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

i.e., find X, Y, Δ_1, and Δ_2 such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}.$$

b) Derive the inverse of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

c) Show that

$$\left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| = |A - BD^{-1}C||D|.$$

Problem 1.2 Let

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix} \right).$$

Show that the conditional pdf $p_{X|Y}(x|y)$ is Gaussian with mean

$$\mu_{x|y} = \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)$$

and covariance

$$\Sigma_{x|y} = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}.$$

Hint: Use the results from Problem 1.1.

*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

Problem 1.3 Suppose X and Y are jointly distributed random variables. When Y is unknown, an appealing estimate of the value of X is its mean value $\mu_x = \mathsf{E}[X]$. This estimate has the property

$$\mathsf{E}\left[\|X-\mu_x\|^2\right] \leq \mathsf{E}\left[\|X-z\|^2\right]$$

for all z, and has the average error $\mathsf{E}\left[\|X-\mu_x\|^2\right]$. Now suppose that one is told that $Y = y$. Let $\hat{x} = \mathsf{E}_{X|Y=y}\left[X\right]$. Show that

$$\mathsf{E}_{X|Y=y}\left[\|X-\hat{x}\|^2\right] \leq \mathsf{E}_{X|Y=y}\left[\|X-\mu_x\|^2\right].$$

This directly gives the intuitively reasonable result that the mean estimation error $\mathsf{E}\left[\|X-\hat{x}\|^2\right]$ averaged over all values of X and Y is be bounded from above by $\mathsf{E}\left[\|X-\mu_x\|^2\right]$. When is this bound attained, *i.e.*, when is there no improvement in the knowledge of X given the value of Y?

Problem 1.4 Given a zero-mean wide-sense stationary process $y(t)$ with *auto correlation function* (acf) $r_{yy}(\tau)$, derive the linear least-squares estimate of

$$\int_0^T y(t)\,dt$$

in terms of its values at the end points $y(0)$ and $y(T)$.

Assuming that the covariance function is

$$r_{yy}(\tau) = e^{-\alpha|t|},$$

show that the estimate is

$$\frac{1}{\alpha}\tanh\frac{\alpha T}{2}\left(y(0)+y(T)\right).$$

Furthermore, provide an approximate estimate that holds for small T.

Problem 1.5 Assume X and Z are independent (jointly) Gaussian random variables with means μ_x and μ_z, respectively, and with covariance matrices Σ_{xx} and Σ_{zz}. Then $Y = X+Z$ is Gaussian and $p_{X|Y}(x|y)$ is a Gaussian density function. Show that the associated conditional mean and covariance are

$$\begin{aligned}\mu_{x|y} &= \Sigma_{zz}(\Sigma_{xx}+\Sigma_{zz})^{-1}\mu_x + \Sigma_{xx}(\Sigma_{xx}+\Sigma_{zz})^{-1}(y-\mu_z)\\ &= \Sigma_{x|y}\left(\Sigma_{xx}^{-1}\mu_x + \Sigma_{zz}^{-1}(y-\mu_z)\right)\end{aligned}$$

and

$$\begin{aligned}\Sigma_{x|y} &= \Sigma_{xx} - \Sigma_{xx}(\Sigma_{xx}+\Sigma_{zz})^{-1}\Sigma_{xx}\\ &= \left(\Sigma_{xx}^{-1}+\Sigma_{zz}^{-1}\right)^{-1},\end{aligned}$$

respectively.

Hint: Begin by deriving the joint density $p_{X,Y}(x,y)$ and assume that the inverses exist.