Optimal Filtering
Exercise 2

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Problem 2.1 Given a sequence of zero-mean random variables $\{y_1, y_1, \dots\}$ let

$$e_i = y_i - \hat{y}_{i|i-1}, \quad \hat{y}_{1|0} = 0$$
$$\hat{y}_{i|i-1} = \text{llse. of } y_i \text{ given } y_1, \dots, y_{i-1}.$$

a) Show that the e_i are orthogonal random variables.
Hint: recall Gram-Schmidt.

b) Show that if $\mathsf{E}\left[e_i^2\right] > 0 \ \ \forall i$, then the vectors

$$\mathbf{y}_n = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad \mathbf{e}_n = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

are related for all n by a non-singular triangular matrix $\mathbf{T}_n$, *i.e.*, $\mathbf{y}_n = \mathbf{T}_n \mathbf{e}_n$.

c) If $\mathbf{H}_n$ is a linear operator that yields $\hat{\mathbf{y}}_n$ from $\mathbf{y}_n$, show that

$$\mathbf{H}_n = \mathbf{I} - \mathbf{T}_n^{-1}, \qquad \hat{\mathbf{y}}_n = \mathbf{y}_n - \mathbf{e}_n.$$

d) Let $r_{ye}(i,j) = \mathsf{E}\{y_i e_j^*\}$. Show that

$$\mathbf{T}_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ r_{ye}(2,1) r_{ee}^{-1}(1,1) & 1 & 0 & \dots & 0 \\ r_{ye}(3,1) r_{ee}^{-1}(1,1) & r_{ye}(3,2) r_{ee}^{-1}(2,2) & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{ye}(n,1) r_{ee}^{-1}(1,1) & r_{ye}(n,2) r_{ee}^{-1}(2,2) & \dots & r_{ye}(n,n-1) r_{ee}^{-1}(n-1,n-1) & 1 \end{pmatrix}.$$

e) Show that $\mathbf{R}_{yy} = \mathbf{T}_n \mathbf{R}_{ee} \mathbf{T}_n^*$, where $\mathbf{R}_{yy} = \mathsf{E}\left[\mathbf{y}_n \mathbf{y}_n^*\right]$ and $\mathbf{R}_{ee} = \mathsf{E}\left[\mathbf{e}_n \mathbf{e}_n^*\right]$.

f) Let $\mathbf{x}$ be a zero-mean stochastic vector related to the random process y_i and let $\hat{\mathbf{x}}_n$ denote the llse. of $\mathbf{x}$ based on the observations of y_i up to $i = n$. Show that

$$\hat{\mathbf{x}}_n = \mathbf{R}_{xy} \mathbf{R}_{yy}^{-1} \mathbf{y}_n = \mathbf{R}_{xe} \mathbf{R}_{ee}^{-1} \mathbf{e}_n.$$

*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

g) Show that

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \mathsf{E}\left[\mathbf{x}e_{n+1}^*\right] r_{ee}^{-1}(n+1,n+1)e_{n+1}.$$

h) Reflect on the similarity between the results in e) and f) with the derivation of the causal Wiener filter in Lecture 2.

Problem 2.2 Let $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$, where

$$\mathsf{E}\left[\mathbf{x}\mathbf{x}^*\right] = \mathbf{\Pi}, \qquad \mathsf{E}\left[\mathbf{x}\mathbf{v}^*\right] = \mathbf{0}, \qquad \mathsf{E}\left[\mathbf{v}\mathbf{v}^*\right] = \mathbf{R}_{vv}, \qquad \mathsf{E}\left[\mathbf{y}\mathbf{y}^*\right] = \mathbf{R}_{yy}.$$

a) Show that the llse. of $\mathbf{x}$ can be written

$$\hat{\mathbf{x}} = \mathbf{\Pi}\mathbf{H}^*\left(\mathbf{R}_{vv} + \mathbf{H}\mathbf{\Pi}\mathbf{H}^*\right)^{-1}\mathbf{y}.$$

b) If $|\mathbf{R}_{vv}| \neq 0$ and $|\mathbf{\Pi}| \neq 0$, show that

$$\hat{\mathbf{x}} = \left(\mathbf{\Pi}^{-1} + \mathbf{H}^*\mathbf{R}_{vv}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^*\mathbf{R}_{vv}^{-1}\mathbf{y}.$$

Problem 2.3 Let x be a zero-mean non-Gaussian random variable with moments $\mathrm{E}\{x^n\} = \nu_n$.

a) Find the *linear least squares estimate* (llse.) of x^3 given x.

Note: Here *linear* also includes affine transformations.

b) Find the *least squares estimate* (lse.) of x^3 given x.

Problem 2.4 Consider the Wiener filtering problem of estimating x_{k+n} from $\{y_i\}_{i=-\infty}^{k}$.

a) Show that

$$\begin{aligned}
\min \mathrm{mse} &= \mathsf{E}\left[x_{k+n}^2\right] - \mathsf{E}\left[x_{k+n}\hat{x}_{k+n|k}\right] \\
&= r_{xx}(0) - \mathsf{E}\left[\hat{x}_{k+n}^2\right] \\
&= r_{xx}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left[\frac{\Phi_{xy}(e^{j\omega})e^{j\omega n}}{T(e^{-j\omega})}\right]_+\right|^2 d\omega \\
&= r_{xx}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|\Phi_{xy}(e^{j\omega})\right|^2}{\Phi_{yy}(e^{j\omega})} d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left[\frac{\Phi_{xy}(e^{j\omega})e^{j\omega n}}{T(e^{-j\omega})}\right]_-\right|^2 d\omega.
\end{aligned}$$

Here $\Phi_{yy}(e^{j\omega})$ ($\Phi_{xy}(e^{j\omega})$) denotes the spectrum (cross-spectrum) and the spectrum has the spectral factorization $\Phi_{yy}(e^{j\omega}) = T(e^{j\omega})T^*(e^{-j\omega})$.

Note: If you are browsing for hints in the book, there is a typo on page 257, where the hint in 7.19(f) should say

$$\int_{-\pi}^{\pi}\left|a(e^{j\omega})\right|^2 d\omega = \int_{-\pi}^{\pi}\left|\left[a(e^{j\omega})\right]_+\right|^2 d\omega + \int_{-\pi}^{\pi}\left|\left[a(e^{j\omega})\right]_-\right|^2 d\omega.$$

If you wish to use this result, first prove that it is true.

b) Interpret the result in a).

Problem 2.5 If

$$\Phi_{yy}(z) = \frac{-\beta z^{-1} + 1 + |\beta|^2 - \beta^* z}{-\alpha z^{-1} + 1 + |\alpha|^2 - \alpha^* z}, \quad |\alpha| < 1, \quad |\beta| > 1,$$

show that the causal Wiener predictor for y_{k+n} given observations $\{y_i\}_{i=-\infty}^{k}$ is

$$\hat{y}_{k+n|k} = \left(\alpha - \beta^{-*}\right)\alpha^{n-1} \sum_{i=0}^{\infty} \left(\beta^{-*}\right)^i y_{k-i}$$

and has the MSE

$$\mathsf{E}\left[(y_{k+n} - \hat{y}_{k+n|k})^2\right] = |\beta|^2 + |\alpha\beta^* - 1|^2 \frac{1 - |\alpha|^{2n-2}}{1 - |\alpha|^2}.$$