

Optimal Filtering

Exercise 2

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Problem 2.1 Given a sequence of zero-mean random variables $\{y_1, y_1, \dots\}$ let

$$e_i = y_i - \hat{y}_{i|i-1}, \quad \hat{y}_{1|0} = 0$$

$$\hat{y}_{i|i-1} = \text{llse. of } y_i \text{ given } y_1, \dots, y_{i-1}.$$

a) Show that the e_i are orthogonal random variables.

Hint: recall Gram-Schmidt.

b) Show that if $E[e_i^2] > 0 \forall i$, then the vectors

$$\mathbf{y}_n = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad \mathbf{e}_n = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

are related for all n by a non-singular triangular matrix \mathbf{T}_n , i.e., $\mathbf{y}_n = \mathbf{T}_n \mathbf{e}_n$.

c) If \mathbf{H}_n is a linear operator that yields $\hat{\mathbf{y}}_n$ from \mathbf{y}_n , show that

$$\mathbf{H}_n = \mathbf{I} - \mathbf{T}_n^{-1}, \quad \hat{\mathbf{y}}_n = \mathbf{y}_n - \mathbf{e}_n.$$

d) Let $r_{ye}(i, j) = E\{y_i e_j^*\}$. Show that

$$\mathbf{T}_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ r_{ye}(2,1)r_{ee}^{-1}(1,1) & 1 & 0 & \dots & 0 \\ r_{ye}(3,1)r_{ee}^{-1}(1,1) & r_{ye}(3,2)r_{ee}^{-1}(2,2) & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{ye}(n,1)r_{ee}^{-1}(1,1) & r_{ye}(n,2)r_{ee}^{-1}(2,2) & \dots & r_{ye}(n,n-1)r_{ee}^{-1}(n-1,n-1) & 1 \end{pmatrix}.$$

e) Show that $\mathbf{R}_{yy} = \mathbf{T}_n \mathbf{R}_{ee} \mathbf{T}_n^*$, where $\mathbf{R}_{yy} = E[\mathbf{y}_n \mathbf{y}_n^*]$ and $\mathbf{R}_{ee} = E[\mathbf{e}_n \mathbf{e}_n^*]$.

f) Let \mathbf{x} be a zero-mean stochastic vector related to the random process y_i and let $\hat{\mathbf{x}}_n$ denote the llse. of \mathbf{x} based on the observations of y_i up to $i = n$. Show that

$$\hat{\mathbf{x}}_n = \mathbf{R}_{xy} \mathbf{R}_{yy}^{-1} \mathbf{y}_n = \mathbf{R}_{xe} \mathbf{R}_{ee}^{-1} \mathbf{e}_n.$$

*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

g) Show that

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \mathbf{E}[\mathbf{x}e_{n+1}^*]r_{ee}^{-1}(n+1, n+1)e_{n+1}.$$

h) Reflect on the similarity between the results in e) and f) with the derivation of the causal Wiener filter in Lecture 2.

Problem 2.2 Let $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$, where

$$\mathbf{E}[\mathbf{x}\mathbf{x}^*] = \mathbf{\Pi}, \quad \mathbf{E}[\mathbf{x}\mathbf{v}^*] = \mathbf{0}, \quad \mathbf{E}[\mathbf{v}\mathbf{v}^*] = \mathbf{R}_{vv}, \quad \mathbf{E}[\mathbf{y}\mathbf{y}^*] = \mathbf{R}_{yy}.$$

a) Show that the llse. of \mathbf{x} can be written

$$\hat{\mathbf{x}} = \mathbf{\Pi}\mathbf{H}^*(\mathbf{R}_{vv} + \mathbf{H}\mathbf{\Pi}\mathbf{H}^*)^{-1}\mathbf{y}.$$

b) If $|\mathbf{R}_{vv}| \neq 0$ and $|\mathbf{\Pi}| \neq 0$, show that

$$\hat{\mathbf{x}} = (\mathbf{\Pi}^{-1} + \mathbf{H}^*\mathbf{R}_{vv}^{-1}\mathbf{H})^{-1}\mathbf{H}^*\mathbf{R}_{vv}^{-1}\mathbf{y}.$$

Problem 2.3 Let x be a zero-mean non-Gaussian random variable with moments $\mathbf{E}\{x^n\} = v_n$.

a) Find the *linear least squares estimate* (llse.) of x^3 given x .

Note: Here *linear* also includes affine transformations.

b) Find the *least squares estimate* (lse.) of x^3 given x .

Problem 2.4 Consider the Wiener filtering problem of estimating x_{k+n} from $\{y_i\}_{i=-\infty}^k$.

a) Show that

$$\begin{aligned} \min \text{mse} &= \mathbf{E}[x_{k+n}^2] - \mathbf{E}[x_{k+n}\hat{x}_{k+n|k}] \\ &= r_{xx}(0) - \mathbf{E}[\hat{x}_{k+n}^2] \\ &= r_{xx}(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \left[\frac{\Phi_{xy}(e^{j\omega})e^{j\omega n}}{T(e^{-j\omega})} \right]_+ \right|^2 d\omega \\ &= r_{xx}(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\Phi_{xy}(e^{j\omega})|^2}{\Phi_{yy}(e^{j\omega})} d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \left[\frac{\Phi_{xy}(e^{j\omega})e^{j\omega n}}{T(e^{-j\omega})} \right]_- \right|^2 d\omega. \end{aligned}$$

Here $\Phi_{yy}(e^{j\omega})$ ($\Phi_{xy}(e^{j\omega})$) denotes the spectrum (cross-spectrum) and the spectrum has the spectral factorization $\Phi_{yy}(e^{j\omega}) = T(e^{j\omega})T^*(e^{-j\omega})$.

Note: If you are browsing for hints in the book, there is a typo on page 257, where the hint in 7.19(f) should say

$$\int_{-\pi}^{\pi} |a(e^{j\omega})|^2 d\omega = \int_{-\pi}^{\pi} |[a(e^{j\omega})]_+|^2 d\omega + \int_{-\pi}^{\pi} |[a(e^{j\omega})]_-|^2 d\omega.$$

If you wish to use this result, first prove that it is true.

b) Interpret the result in a).

Problem 2.5 If

$$\Phi_{yy}(z) = \frac{-\beta z^{-1} + 1 + |\beta|^2 - \beta^* z}{-\alpha z^{-1} + 1 + |\alpha|^2 - \alpha^* z}, \quad |\alpha| < 1, \quad |\beta| > 1,$$

show that the causal Wiener predictor for y_{k+n} given observations $\{y_i\}_{i=-\infty}^k$ is

$$\hat{y}_{k+n|k} = (\alpha - \beta^{-*}) \alpha^{n-1} \sum_{i=0}^{\infty} (\beta^{-*})^i y_{k-i}$$

and has the MSE

$$E[(y_{k+n} - \hat{y}_{k+n|k})^2] = |\beta|^2 + |\alpha \beta^* - 1|^2 \frac{1 - |\alpha|^{2n-2}}{1 - |\alpha|^2}.$$