

Optimal Filtering

Exercise 3

Gustaf Hendeby*
gustaf.hendeby@liu.se

Version: 2023-11-11

Problem 3.1 Assume the following measurement function

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k,$$

where \mathbf{v}_k is zero-mean Gaussian with covariance \mathbf{R}_k . Show that if \mathbf{R}_k is singular, a matrix \mathbf{A} exists such that $\mathbf{A}\mathbf{y}_k$ is a deterministic function of \mathbf{x}_k . What is the maximum rank of \mathbf{A} for which this property holds?

Problem 3.2 Consider the standard linear state-space model

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k & \text{cov}(\mathbf{w}_k) &= \mathbf{Q}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k & \text{cov}(\mathbf{v}_k) &= \mathbf{R}_k, \end{aligned}$$

with $\mathbf{S}_k = \mathbf{0}$ and assume $\mathbf{\Pi}_0 \succ \mathbf{0}$.

a) Make necessary assumptions and show that the covariance of a two-step predictor is given by

$$\mathbf{P}_{k+1|k-1} = \mathbf{F}_k \mathbf{P}_{k|k-1} \mathbf{F}_k^* + \mathbf{G}_k \mathbf{Q}_k \mathbf{G}_k^*$$

b) Find the relationship between $\tilde{\mathbf{x}}_{k+1|k-1}$ and $\tilde{\mathbf{x}}_{k+1|k}$, where $\tilde{\mathbf{x}}_{i|j} = \mathbf{x}_i - \hat{\mathbf{x}}_{i|j}$, and use it to show that

$$\mathbf{P}_{k+1|k-1} = \mathbf{P}_{k+1|k} + \mathbf{F}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^* (\mathbf{R}_k + \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^*)^{-1} \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{F}_k^*$$

Problem 3.3 The matrix $\mathbf{P}_{k+1|k}$ is per definition non-negative definite symmetric because it is a covariance matrix. Show by induction arguments that the Kalman filter equations for calculating $\mathbf{P}_{k+1|k}$ imply (algebraically) that $\mathbf{P}_{k+1|k}$ is nonnegative definite symmetric.

*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

Problem 3.4 Your task is to derive a Kalman based solution to estimate the initial state value \mathbf{x}_0 given the measurements $\{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_k\}$. Assume the standard linear state-space model

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k & \text{cov}(\mathbf{w}_k) &= \mathbf{Q}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k & \text{cov}(\mathbf{v}_k) &= \mathbf{R}_k,\end{aligned}$$

with $\mathbf{S}_k = \mathbf{0}$.

Show that the resulting estimate is given by the standard Kalman filter for \mathbf{x}_k extended with the following equations

$$\begin{aligned}\mathbf{K}_k^0 &= \mathbf{P}_k^0 \mathbf{H}_k^* (\mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^* + \mathbf{R}_k)^{-1} \\ \hat{\mathbf{x}}_{0|k} &= \hat{\mathbf{x}}_k^0 = \hat{\mathbf{x}}_{k-1}^0 + \mathbf{K}_k^0 (\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}) \\ \mathbf{P}_{k+1}^0 &= \mathbf{P}_k^0 \mathbf{F}_k^* - \mathbf{K}_k^0 \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{F}_k^* \\ \mathbf{P}_{k+1}^{00} &= \mathbf{P}_k^{00} - \mathbf{K}_k^0 \mathbf{H}_k (\mathbf{P}_k^0)^*\end{aligned}$$

which is initialized by

$$\begin{aligned}\hat{\mathbf{x}}_{-1}^0 &= \hat{\mathbf{x}}_{0|-1} \\ \mathbf{P}_0^{00} &= \mathbf{P}_0^0 = \mathbf{P}_{0|-1}\end{aligned}$$

Furthermore, explain what \mathbf{P}_k^0 and \mathbf{P}_k^{00} denote and represent conceptually.

Hint: Introduce an extended state vector $\begin{pmatrix} \mathbf{x}_k \\ \mathbf{x}_k^0 \end{pmatrix}$, where $\mathbf{x}_{k+1}^0 = \mathbf{x}_k^0$, $\forall k \geq 0$, in addition to the standard state-space model. To match the results above, study the one-step Kalman predictions.

Problem 3.5 Filtered residuals $\boldsymbol{\mu}_k$ are defined as

$$\begin{aligned}\mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k & \text{cov}(\mathbf{v}_k) &= \mathbf{R}_k \\ \boldsymbol{\mu}_k &= \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k}.\end{aligned}$$

Furthermore, define $\mathbf{e}_k = \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}$.

Show that

- $\boldsymbol{\mu}_k = \mathbf{R}_k \mathbf{R}_{e_k}^{-1} \mathbf{e}_k$
- $\{\boldsymbol{\mu}_k\}$ is a white sequence with covariance matrix $\mathbf{R}_k^\mu = \mathbf{R}_k - \mathbf{H}_k \mathbf{P}_{k|k} \mathbf{H}_k^*$.