Optimal Filtering Exercise 3

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Problem 3.1 Assume the following measurement function

 $\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k,$

where \mathbf{v}_k is zero-mean Gaussian with covariance \mathbf{R}_k . Show that if \mathbf{R}_k is singular, a matrix *A* exists such that $\mathbf{A}\mathbf{y}_k$ is a deterministic function of \mathbf{x}_k . What is the maximum rank of **A** for which this property holds?

Problem 3.2 Consider the standard linear state-space model

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k & \operatorname{cov}(\mathbf{w}_k) = \mathbf{Q}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k & \operatorname{cov}(\mathbf{v}_k) = \mathbf{R}_k, \end{aligned}$$

with $\mathbf{S}_k = \mathbf{0}$ and assume $\mathbf{\Pi}_0 \succ 0$.

a) Make necessary assumptions and show that the covariance of a two-step predictor is given by

$$\mathbf{P}_{k+1|k-1} = \mathbf{F}_k \mathbf{P}_{k|k-1} \mathbf{F}_k^* + \mathbf{G}_k \mathbf{Q}_k \mathbf{G}_k^*$$

b) Find the relationship between $\tilde{\mathbf{x}}_{k+1|k-1}$ and $\tilde{\mathbf{x}}_{k+1|k}$, where $\tilde{\mathbf{x}}_{i|j} = \mathbf{x}_i - \hat{\mathbf{x}}_{i|j}$, and use it to show that

$$\mathbf{P}_{k+1|k-1} = \mathbf{P}_{k+1|k} + \mathbf{F}_{k}\mathbf{P}_{k|k-1}\mathbf{H}_{k}^{*}(\mathbf{R}_{k} + \mathbf{H}_{k}\mathbf{P}_{k|k-1}\mathbf{H}_{k}^{*})^{-1}\mathbf{H}_{k}\mathbf{P}_{k|k-1}\mathbf{F}_{k}^{*}$$

Problem 3.3 The matrix $\mathbf{P}_{k+1|k}$ is per definition non-negative definite symmetric because it is a covariance matrix. Show by induction arguments that the Kalman filter equations for calculating $\mathbf{P}_{k+1|k}$ imply (algebraically) that $\mathbf{P}_{k+1|k}$ is nonnegative definite symmetric.

^{*}These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

Problem 3.4 Your task is to derive a Kalman based solution to estimate the initial state value \mathbf{x}_0 given the measurements $\{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_k\}$. Assume the standard linear state-space model

$$\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k \qquad \qquad \mathsf{cov}(\mathbf{w}_k) = \mathbf{Q}_k \\ \mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \qquad \qquad \qquad \mathsf{cov}(\mathbf{v}_k) = \mathbf{R}_k ;$$

with $\mathbf{S}_k = \mathbf{0}$.

Show that the resulting estimate is given by the standard Kalman filter for \mathbf{x}_k extended with the following equations

$$\mathbf{K}_{k}^{0} = \mathbf{P}_{k}^{0}\mathbf{H}_{k}^{*}(\mathbf{H}_{k}\mathbf{P}_{k|k-1}\mathbf{H}_{k}^{*} + \mathbf{R}_{k})^{-1}$$
$$\hat{\mathbf{x}}_{0|k} = \hat{\mathbf{x}}_{k}^{0} = \hat{\mathbf{x}}_{k-1}^{0} + \mathbf{K}_{k}^{0}(\mathbf{y}_{k} - \mathbf{H}_{k}\hat{\mathbf{x}}_{k|k-1})$$
$$\mathbf{P}_{k+1}^{0} = \mathbf{P}_{k}^{0}\mathbf{F}_{k}^{*} - \mathbf{K}_{k}^{0}\mathbf{H}_{k}\mathbf{P}_{k|k-1}\mathbf{F}_{k}^{*}$$
$$\mathbf{P}_{k+1}^{00} = \mathbf{P}_{k}^{00} - \mathbf{K}_{k}^{0}\mathbf{H}_{k}(\mathbf{P}_{k}^{0})^{*}$$

which is initialized by

$$\hat{\mathbf{x}}_{-1}^{0} = \hat{\mathbf{x}}_{0|-1}$$
$$\mathbf{P}_{0}^{00} = \mathbf{P}_{0}^{0} = \mathbf{P}_{0|-1}$$

Furthermore, explain what \mathbf{P}_k^0 and \mathbf{P}_k^{00} denote and represent conceptually.

Hint: Introduce an extended state vector $\begin{pmatrix} \mathbf{x}_k \\ \mathbf{x}_k^0 \end{pmatrix}$, where $\mathbf{x}_{k+1}^0 = \mathbf{x}_k^0$, $\forall k \ge 0$, in addition to the standard state-space model. To match the results above, study the one-step Kalman predictions.

Problem 3.5 Filtered residuals μ_k are defined as

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k$$
 $\operatorname{cov}(\mathbf{v}_k) = R_k$
 $\boldsymbol{\mu}_k = \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k}.$

Furthermore, define $\mathbf{e}_k = \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}$.

Show that

- a) $\boldsymbol{\mu}_k = \mathbf{R}_k \mathbf{R}_{e_k}^{-1} \mathbf{e}_k$
- **b**) $\{\boldsymbol{\mu}_k\}$ is a white sequence with covariance matrix $\mathbf{R}_k^{\mu} = \mathbf{R}_k \mathbf{H}_k \mathbf{P}_{k|k} \mathbf{H}_k^*$.