Optimal Filtering
Exercise 3

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Problem 3.1 Assume the following measurement function

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k,$$

where $\mathbf{v}_k$ is zero-mean Gaussian with covariance $\mathbf{R}_k$. Show that if $\mathbf{R}_k$ is singular, a matrix A exists such that $\mathbf{A}\mathbf{y}_k$ is a deterministic function of $\mathbf{x}_k$. What is the maximum rank of $\mathbf{A}$ for which this property holds?

Problem 3.2 Consider the standard linear state-space model

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k & \qquad \operatorname{cov}(\mathbf{w}_k) &= \mathbf{Q}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k & \qquad \operatorname{cov}(\mathbf{v}_k) &= \mathbf{R}_k, \end{aligned}$$

with $\mathbf{S}_k = \mathbf{0}$ and assume $\mathbf{\Pi}_0 \succ 0$.

a) Make necessary assumptions and show that the covariance of a two-step predictor is given by

$$\mathbf{P}_{k+1|k-1} = \mathbf{F}_k \mathbf{P}_{k|k-1} \mathbf{F}_k^* + \mathbf{G}_k \mathbf{Q}_k \mathbf{G}_k^*$$

b) Find the relationship between $\tilde{\mathbf{x}}_{k+1|k-1}$ and $\tilde{\mathbf{x}}_{k+1|k}$, where $\tilde{\mathbf{x}}_{i|j} = \mathbf{x}_i - \hat{\mathbf{x}}_{i|j}$, and use it to show that

$$\mathbf{P}_{k+1|k-1} = \mathbf{P}_{k+1|k} + \mathbf{F}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^* \big(\mathbf{R}_k + \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^*\big)^{-1} \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{F}_k^*$$

Problem 3.3 The matrix $\mathbf{P}_{k+1|k}$ is per definition non-negative definite symmetric because it is a covariance matrix. Show by induction arguments that the Kalman filter equations for calculating $\mathbf{P}_{k+1|k}$ imply (algebraically) that $\mathbf{P}_{k+1|k}$ is nonnegative definite symmetric.

*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

Problem 3.4 Your task is to derive a Kalman based solution to estimate the initial state value $\mathbf{x}_0$ given the measurements $\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k\}$. Assume the standard linear state-space model

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{F}_k\mathbf{x}_k + \mathbf{G}_k\mathbf{w}_k & \mathsf{cov}(\mathbf{w}_k) &= \mathbf{Q}_k \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k & \mathsf{cov}(\mathbf{v}_k) &= \mathbf{R}_k, \end{aligned}$$

with $\mathbf{S}_k = \mathbf{0}$.

Show that the resulting estimate is given by the standard Kalman filter for $\mathbf{x}_k$ extended with the following equations

$$\begin{aligned} \mathbf{K}_k^0 &= \mathbf{P}_k^0\mathbf{H}_k^*\left(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^* + \mathbf{R}_k\right)^{-1} \\ \hat{\mathbf{x}}_{0|k} &= \hat{\mathbf{x}}_k^0 = \hat{\mathbf{x}}_{k-1}^0 + \mathbf{K}_k^0\left(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k|k-1}\right) \\ \mathbf{P}_{k+1}^0 &= \mathbf{P}_k^0\mathbf{F}_k^* - \mathbf{K}_k^0\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{F}_k^* \\ \mathbf{P}_{k+1}^{00} &= \mathbf{P}_k^{00} - \mathbf{K}_k^0\mathbf{H}_k(\mathbf{P}_k^0)^* \end{aligned}$$

which is initialized by

$$\begin{aligned} \hat{\mathbf{x}}_{-1}^0 &= \hat{\mathbf{x}}_{0|-1} \\ \mathbf{P}_0^{00} &= \mathbf{P}_0^0 = \mathbf{P}_{0|-1} \end{aligned}$$

Furthermore, explain what $\mathbf{P}_k^0$ and $\mathbf{P}_k^{00}$ denote and represent conceptually.

Hint: Introduce an extended state vector $\begin{pmatrix}\mathbf{x}_k \\ \mathbf{x}_k^0\end{pmatrix}$, where $\mathbf{x}_{k+1}^0 = \mathbf{x}_k^0,\ \forall k \geq 0$, in addition to the standard state-space model. To match the results above, study the one-step Kalman predictions.

Problem 3.5 Filtered residuals $\boldsymbol{\mu}_k$ are defined as

$$\begin{aligned} \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k & \mathsf{cov}(\mathbf{v}_k) &= R_k \\ \boldsymbol{\mu}_k &= \mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k|k}. \end{aligned}$$

Furthermore, define $\mathbf{e}_k = \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}$.

Show that

a) $\boldsymbol{\mu}_k = \mathbf{R}_k\mathbf{R}_{e_k}^{-1}\mathbf{e}_k$

b) $\{\boldsymbol{\mu}_k\}$ is a white sequence with covariance matrix $\mathbf{R}_k^{\mu} = \mathbf{R}_k - \mathbf{H}_k\mathbf{P}_{k|k}\mathbf{H}_k^*$.