Optimal Filtering
Exercise 4

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Problem 4.1 Consider the state-space model with a direct feed-through

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k + \mathbf{J}_k \mathbf{w}_k,\end{aligned}$$

where $\mathbf{v}_k$ and $\mathbf{w}_k$ are independent, zero-mean, white Gaussian processes with covariance matrices $\mathbf{R}_k$ and $\mathbf{Q}_k$, respectively. Show that this arrangement is equivalent to one with $\mathbf{y} = \mathbf{H}_k \mathbf{x}_k + \bar{\mathbf{v}}_k$. Characterize $\bar{\mathbf{v}}_k$.

Problem 4.2

a) Consider the signal model in Problem 4.1. Let $\mathbf{R}_k = 0\ \forall k$, and assume that $\mathbf{x}_0 = \mathbf{0}$. Show that if $\mathbf{J}_k$ is non-singular, $\mathbf{P}_{k|k-1} = \mathbf{0}\ \forall k$.

b) Explain why this should be so.

Problem 4.3 Consider the scalar state-space model

$$\begin{aligned}x_{k+1} &= x_k + w_k \\ y_k &= x_k + v_k,\end{aligned}$$

where w_k and v_k are mutually independent white noise processes with variance 1. Compare the performance of the Kalman filter and the one-step back Kalman smoother, by analytically determining the ratio

$$\frac{\lim_{k\to\infty} \mathsf{E}\left[\left(x_k - \hat{x}_{k|k+1}\right)^2\right]}{\lim_{k\to\infty} \mathsf{E}\left[\left(x_k - \hat{x}_{k|k}\right)^2\right]}.$$

Comment on the result.

*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

Problem 4.4 Consider the scalar state-space model

$$
\begin{aligned}
x_{k+1} &= 0.8x_k + w_k, \\
y_k &= x_k + v_k,
\end{aligned}
$$

where w_k and v_k are independent white noise processes with variance $\sigma_w^2 = 0.6^2$ and $\sigma_v^2 = 1$, respectively.

a) Plot the frequency and phase response of the stationary one-step ahead Kalman filter predictor for x_k.

b) Plot the frequency and phase response of the one-step ahead causal Wiener filter predictor for x_k.

c) Comment on the result.

Problem 4.5 *Bonus problem! This problem does not count towards the maximum score of this homework! (That is, the total score for homework 4 is only 40 points.) However, any points on this exercise counts towards your total score as bonus points.*

Consider the state-space model

$$
\begin{aligned}
\mathbf{x}_{k+} &= \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{w}_k \\
\mathbf{y}_k &= \mathbf{H}\mathbf{x}_k + \mathbf{v}_k,
\end{aligned}
$$

where

$$
\mathsf{E}\left[\begin{pmatrix} \mathbf{w}_k \\ \mathbf{v}_k \\ \mathbf{x}_0 \end{pmatrix} \begin{bmatrix} \mathbf{w}_l \\ \mathbf{v}_l \\ \mathbf{x}_0 \\ \mathbf{1} \end{bmatrix}^*\right] = \begin{pmatrix} \mathbf{Q}\delta_{k-l} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}\delta_{k-l} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Pi}_0 & \mathbf{0} \end{pmatrix}
$$

and $|\lambda_{\max}(\mathbf{F})| < 1$.

a) Let $\bar{\mathbf{P}} = \lim_{k\to\infty} \mathbf{P}_{k|k-1}$ and $\bar{\mathbf{\Pi}} = \lim_{k\to\infty} \mathsf{E}\left[\mathbf{x}_k\mathbf{x}_k^*\right]$. Show that $\bar{\mathbf{P}} \preceq \bar{\mathbf{\Pi}}$ and comment on the result.

b) A system is called *estimable* if $\bar{\mathbf{P}} \prec \bar{\mathbf{\Pi}}$. Give a one-sentence interpretation of estimablility.

c) Show that the given linear system is *not* estimable if, and only if, there exists some vector $\mathbf{a}$ such that $\mathbf{a}^*\hat{\mathbf{x}}_{k|k-1} = \mathbf{0}\ \forall k$.

d) Show that the given linear system is *not* estimable if, and only if, there exists some vector $\mathbf{a}$ such that $\mathbf{a}^*\mathbf{x}_k \perp \mathscr{L}\big(\mathbf{Y}_{k-1}\big)$.

Hint: See the paper
Y. Baram and T. Kailath, “Estimability and regulability of linear systems,” *IEEE Transactions on Automatic Control*, vol. 33, no. 12, pp. 1116–1121, 1988.