

# Optimal Filtering

## Exercise 4

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Version: 2023-11-16

**Problem 4.1** Consider the state-space model with a direct feed-through

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k + \mathbf{J}_k \mathbf{w}_k,\end{aligned}$$

where  $\mathbf{v}_k$  and  $\mathbf{w}_k$  are independent, zero-mean, white Gaussian processes with covariance matrices  $\mathbf{R}_k$  and  $\mathbf{Q}_k$ , respectively. Show that this arrangement is equivalent to one with  $\mathbf{y} = \mathbf{H}_k \mathbf{x}_k + \bar{\mathbf{v}}_k$ . Characterize  $\bar{\mathbf{v}}_k$ .

**Problem 4.2**

- Consider the signal model in Problem 4.1. Let  $\mathbf{R}_k = 0 \forall k$ , and assume that  $\mathbf{x}_0 = \mathbf{0}$ . Show that if  $\mathbf{J}_k$  is non-singular,  $\mathbf{P}_{k|k-1} = \mathbf{0} \forall k$ .
- Explain why this should be so.

**Problem 4.3** Consider the scalar state-space model

$$\begin{aligned}x_{k+1} &= x_k + w_k \\ y_k &= x_k + v_k,\end{aligned}$$

where  $w_k$  and  $v_k$  are mutually independent white noise processes with variance 1. Compare the performance of the Kalman filter and the one-step back Kalman smoother, by analytically determining the ratio

$$\frac{\lim_{k \rightarrow \infty} \mathbb{E}[(x_k - \hat{x}_{k|k+1})^2]}{\lim_{k \rightarrow \infty} \mathbb{E}[(x_k - \hat{x}_{k|k})^2]}.$$

Comment on the result.

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\*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

**Problem 4.4** Consider the scalar state-space model

$$\begin{aligned}x_{k+1} &= 0.8x_k + w_k, \\y_k &= x_k + v_k,\end{aligned}$$

where  $w_k$  and  $v_k$  are independent white noise processes with variance  $\sigma_w^2 = 0.6^2$  and  $\sigma_v^2 = 1$ , respectively.

- Plot the frequency and phase response of the stationary one-step ahead Kalman filter predictor for  $x_k$ .
- Plot the frequency and phase response of the one-step ahead causal Wiener filter predictor for  $x_k$ .
- Comment on the result.

**Problem 4.5** *Bonus problem! This problem does not count towards the maximum score of this homework! (That is, the total score for homework 4 is only 40 points.) However, any points on this exercise counts towards your total score as bonus points.*

Consider the state-space model

$$\begin{aligned}\mathbf{x}_{k+} &= \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{w}_k \\ \mathbf{y}_k &= \mathbf{H}\mathbf{x}_k + \mathbf{v}_k,\end{aligned}$$

where

$$\mathbb{E} \begin{bmatrix} \begin{pmatrix} \mathbf{w}_k \\ \mathbf{v}_k \\ \mathbf{x}_0 \end{pmatrix} \begin{bmatrix} \mathbf{w}_l \\ \mathbf{v}_l \\ \mathbf{x}_0 \\ \mathbf{1} \end{bmatrix}^* \end{bmatrix} = \begin{pmatrix} \mathbf{Q}\delta_{k-l} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}\delta_{k-l} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Pi}_0 & \mathbf{0} \end{pmatrix}$$

and  $|\lambda_{\max}(\mathbf{F})| < 1$ .

- Let  $\bar{\mathbf{P}} = \lim_{k \rightarrow \infty} \mathbf{P}_{k|k-1}$  and  $\bar{\mathbf{\Pi}} = \lim_{k \rightarrow \infty} \mathbb{E}[\mathbf{x}_k \mathbf{x}_k^*]$ . Show that  $\bar{\mathbf{P}} \preceq \bar{\mathbf{\Pi}}$  and comment on the result.
- A system is called *estimable* if  $\bar{\mathbf{P}} \prec \bar{\mathbf{\Pi}}$ . Give a one-sentence interpretation of estimability.
- Show that the given linear system is *not* estimable if, and only if, there exists some vector  $\mathbf{a}$  such that  $\mathbf{a}^* \hat{\mathbf{x}}_{k|k-1} = \mathbf{0} \forall k$ .
- Show that the given linear system is *not* estimable if, and only if, there exists some vector  $\mathbf{a}$  such that  $\mathbf{a}^* \mathbf{x}_k \perp \mathcal{L}(\mathbf{Y}_{k-1})$ .

**Hint:** See the paper

Y. Baram and T. Kailath, "Estimability and regulability of linear systems," *IEEE Transactions on Automatic Control*, vol. 33, no. 12, pp. 1116–1121, 1988.