Optimal Filtering
Exercise 5

Gustaf Hendeby*
gustaf.hendeby@liu.se

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Problem 5.1 Consider the following state-space model

$$\mathbf{x}_{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}_k + \mathbf{w}_k$$
$$\mathbf{y}_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}_k + \mathbf{v}_k$$

where $\mathsf{E}\big[\mathbf{w}_k \mathbf{w}_l^*\big] = \sigma_w^2 \mathbf{I}_2 \delta_{k-l}$, $\mathsf{E}\big[\mathbf{v}_k \mathbf{v}_l^*\big] = \sigma_v^2 \mathbf{I}_2 \delta_{k-l}$, and $\mathsf{E}\big[\mathbf{w}_k \mathbf{v}_l^*\big] = \mathbf{0}_2$.

a) Determine the transfer function $H(z)$ from $\mathbf{y}_k$ to $\hat{\mathbf{x}}_{k+1|k}$ for the stationary Kalman filter.

b) Plot the frequency response of $H(z)$. How does the ratio σ_v^2/σ_w^2 affect the filter characteristics?

Problem 5.2 We wish to estimate the position of an aircraft from noisy radar observations. This is often referred to as a tracking problem. To start with, we will construct a process and measurement model. The aircraft moves with velocity v at an angle θ. We measure r and ϕ.

Let

$$x(n) = \begin{pmatrix} z_1(n) \\ \dot{z}_1(n) \\ z_2(n) \\ \dot{z}_2(n) \end{pmatrix}$$

be the state vector. Assume a constant velocity $\dot{z}_i(n+1) = \dot{z}_i(n)$ and that the time between samples is T.

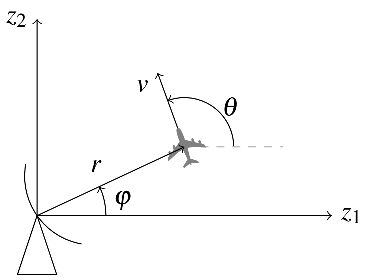

Figure 1: Geometry of the tracking problem.

*These exercises are heavily inspired by exercises used by Isaac Skog (LiU) and Mats Bengtsson (KTH).

a) Derive the state-space model of the scenario in Figure 1, *i.e.*, find the matrices F and H. In the measurement equation, do a variable transformation such that the measurements are z_1 and z_2. The process and measurement noise are assumed independent and white with constant covariance matrices Q and R.

b) We can improve the model by using time-varying covariance matrices. Assume that the disturbances of v and θ are modeled by

$$v(n+1) = v(n) + e_v(n)$$
$$\theta(n+1) = \theta(n) + e_\theta(n),$$

where the noises $e_v(n)$ and $e_\theta(n)$ are independent with variances σ_v^2 and σ_θ^2, respectively.

The measurements are given by

$$\hat{r}(n) = r(n) + e_r(n)$$
$$\hat{\varphi}(n) = \varphi(n) + e_\varphi(n),$$

where $r(n)$ and $\varphi(n)$ are assumed independent, $e_r(n)$ and $e_\theta(n)$ have variances σ_r^2 and σ_θ^2, respectively.

Show that the process noise covariance, Q, can be modeled by:

$$[Q]_{2,2} = \mathsf{E}\left[w_2^2(n)\right] = \sigma_v^2 \cos^2\theta(n) + \sigma_\theta^2 v^2(n) \sin^2\theta(n)$$
$$[Q]_{4,4} = \mathsf{E}\left[w_4^2(n)\right] = \sigma_v^2 \sin^2\theta(n) + \sigma_\theta^2 v^2(n) \cos^2\theta(n)$$
$$[Q]_{2,4} = \mathsf{E}\left[w_2(n)w_4(n)\right] = \left(\sigma_v^2 - v^2(n)\sigma_\theta^2\right)\cos\theta(n)\sin\theta(n)$$

Show that the measurement noise covariance, R, can be modeled by:

$$[R]_{1,1} = \mathsf{E}\left[e_1^2(n)\right] = \sigma_r^2 \cos^2\varphi(n) + \sigma_\varphi^2 r^2(n) \sin^2\varphi(n)$$
$$[R]_{2,2} = \mathsf{E}\left[e_2^2(n)\right] = \sigma_r^2 \sin^2\varphi(n) + \sigma_\varphi^2 r^2(n) \cos^2\varphi(n)$$
$$[R]_{1,2} = \mathsf{E}\left[e_1(n)e_2(n)\right] = \left(\sigma_r^2 - r^2(n)\sigma_\varphi^2\right)\cos\varphi(n)\sin\varphi(n)$$

c) Write scripts (in your language of choice) that plot the states and the outputs, assume the initial state $x(0) = \begin{pmatrix}10 & 1 & 20 & 2\end{pmatrix}^T$. Use $N = 100$ samples. Use the following values for the two cases:

(i) $T = 1$, $Q = 0.1\mathbf{I}$, and $R = 5\mathbf{I}$.

(ii) $T = 1$, $\sigma_v^2 = 0.01$, $\sigma_\theta^2 = 0.01$, $\sigma_r^2 = 1$, and $\sigma_\varphi = 0.1$.

Plot the trajectory and the measured trajectory in the same figure.

Problem 5.3 Design and implement (you are supposed to implement this yourself not simply use an existing toolbox) the Kalman filter associated with the radar problem given above.

a) Utilizing the assumptions made in Problem 5.2a; compare the optimal filter, $K(n)$, to the stationary Kalman filter, $\bar{K}$. (You might want to try with higher noise variance).
Hint: The command `idare` in MATLAB (or similar) might prove useful.

b) Compare the time-varying Kalman filter designed using the result in Problem 5.2b to a Kalman filter using fixed covariance matrices (Problem 5.1a).

In both parts, try the filters with different initial values, $P(0)$ and $x(0)$. How does the choice of $P(0)$ and $x(0)$ affect the behavior?