Lecture #2: LLSE — geometric interpretation

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1 Geometric Interpretation

Recall from Lecture #1, the llse. of x, $\hat{x}$ is given by

$$\hat{x} = \Sigma_{xy}\Sigma_{yy}^{-1} y = K_o y$$

Interpretation:

$$K_o = \Sigma_{xy}\Sigma_{yy}^{-1} \Leftrightarrow K_o \,\mathsf{E}[yy^*] = \mathsf{E}[xy^*] \Leftrightarrow \mathsf{E}[(x - K_o y)y^*] = 0$$

Thus, if the random variables are viewed as vectors, with the inner product defined as $\langle x, y\rangle = \mathsf{E}[xy^*]$, then $\mathsf{E}[(x - K_o y)y^*]$ has the geometric interpretation

$$\langle x - K_o y, y\rangle = 0 \leftrightarrow x - K_o y \perp y,$$

that is, the estimation error is orthogonal to observations!

Two questions related to the geometric interpretation:

1. Which vector space?
   Hilbert space, i.e.*, a complete normed linear vector space.*
2. Is $\langle x, y\rangle = \mathsf{E}[xy^*]$ a proper inner product?
   - Linearity: $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

     $$\mathsf{E}[(\alpha x + \beta y)z^*] = \alpha\,\mathsf{E}[xz^*] + \beta\,\mathsf{E}[yz^*] \quad \textit{Check!}$$

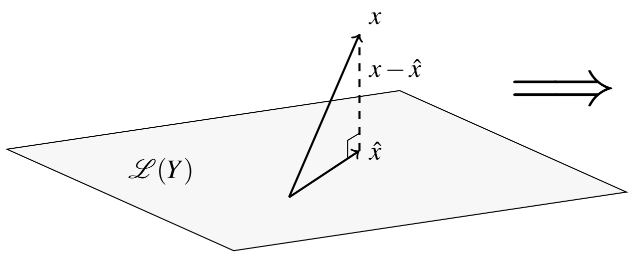
   - Symmetry: $\langle x, y\rangle = \langle y, x\rangle^*$

     $$\mathsf{E}[xy^*] = \mathsf{E}[(yx^*)^*] = \mathsf{E}[yx^*]^* \quad \textit{Check!}$$

   - Non-degeneracy: $\langle x, x\rangle \succeq 0$, *i.e.*, positive semi-definite, and $\langle x, x\rangle = 0$ if and only if $x = 0$.
     $\mathsf{E}[xx^*] = \Sigma_{xx}$*: Covariance matrices are per definition always positive (semi-)definite.*

1.1 Geometric Derivation of LLSE

Find a vector $\hat{x}$ in the linear space spanned by $\{y_1, \dots, y_N\}$ such that $\|x - \hat{x}\|$ is minimized.



$\Longrightarrow$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, \quad \hat{x} = K_o y$$

$$\begin{aligned} x - \hat{x} &\perp Y \\ \langle x - \hat{x}, y \rangle &= 0 \\ \mathsf{E}[xy^*] &= K_o \, \mathsf{E}[yy^*] \\ K_o &= \Sigma_{xy} \Sigma_{yy}^{-1} \end{aligned}$$

2 Winer filtering (WF)

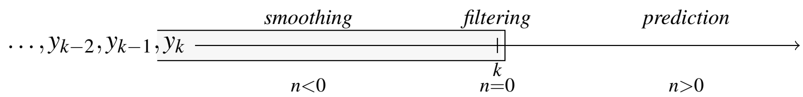
Given the observations $\{y_i\}_{i=-\infty}^{k}$, find the llse. of x_{k+n}.

> **Assumptions:** x_k and y_k are scalar processes, that are jointly stationary with exponentially bounded (cross-)covarainace, *i.e.*, $|r_{xx}(k)| \; < K\alpha^{|k|} \, \forall k$, $K > 0$, $0 < \alpha < 1$, then a spectrum exists.

That is, find

$$\hat{x}_{k+n} = \sum_{i=0}^{\infty} h_{k,i} y_{k-i},$$

where $h_{k,i}$ is possibly time varying filter coefficients subjects to $\mathsf{E}[(x_{k+n} - \hat{x}_{k+n})^2]$ is minimized.



Orthogonality properties gives, for all $j \le k$:

$$\begin{aligned} x_{k+n} - \hat{x}_{k+n} &\perp y_i, \\ \langle x_{k+n} - \sum_{i=0}^{\infty} h_{k,i} y_{k-i}, y_j \rangle &= 0, \\ \mathsf{E}[x_{k+n} y_j^*] &= \sum_{i=0}^{\infty} h_{k,i} \, \mathsf{E}[y_{k-i} y_j^*], \\ r_{xy}(k+n-j) &= \sum_{j=0}^{\infty} h_{k,i} r_{yy}(k-i-j). \end{aligned}$$

Make a change variable, $k - j \to \ell$, gives

$$r_{xy}(\ell + n) = \sum_{i=0}^{\infty} h_{\ell+j,i} r_{yy}(\ell - i), \quad \forall \ell \ge 0.$$

Note, neither $r_{xy}(\ell + n)$ nor $r_{yy}(\ell - i)$ depend on j; hence, $h_{\ell+j,i} = h_i$, *i.e.*, the filter coefficients are time invariant.

Wiener-Hopf equation:

$$r_{xy}(\ell+n) = \sum_{i=a}^{b} h_i r_{yy}(\ell-i), \quad \forall \ell \geq 0.$$

If:
- $\sum_{-\infty}^{\infty}$: Non-causal *Wiener filter* (WF, z-transform), easy to solve.
- $\sum_{a}^{b}$: FIR (*finite impulse response*) WF (linear system equation), easy to solve.
- $\sum_{0}^{\infty}$: Casual WF.
- $\sum_{-\infty}^{-1}$: Anti-casual WF.

2.1 Spectral Factorization

The spectrum of a signal y_k

$$\Phi_{yy}(\omega) = \tilde{\Phi}_{yy}(z)\Big|_{z=e^{i\omega}}$$

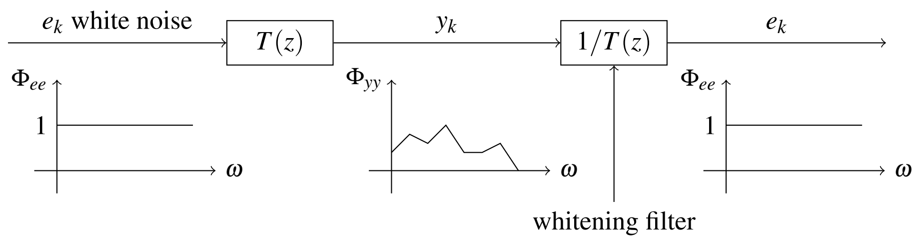
$$\tilde{\Phi}_{yy}(z) = \mathscr{Z}\{\underbrace{r_{yy}(k)}_{\text{acf}}\} = \sum_{k=\infty}^{\infty} r_{yy}(k) z^{-k}$$

Note, $r_{yy}(k) = r_{yy}(-k) \Rightarrow \tilde{\Phi}_{yy}(z) = \tilde{\Phi}_{yy}(z^{-1})$ which implies that $\tilde{\Phi}_{yy}(z)$ has symmetry with respect to mirroring in the unit circle. Hence, if $z = r_i$ has a pole (zero) in the unit circle, then $z = r_i^{-1}$ is also a pole (zero).

If $\tilde{\Phi}_{yy}(z)$ has no poles of zeroes on the unit circle, *i.e.*, $0 < \Phi_{yy}(\omega) < \infty \; \forall \omega$, then

$$\tilde{\Phi}_{yy}(z) = \underbrace{\sigma_e \frac{\prod_{i=1}^{m}(z-r_i)}{\prod_{j=1}^{b}(z-p_j)}}_{=T(z),\text{ stable, causal}} \cdot \underbrace{\sigma_e \frac{\prod_{i=1}^{m}(z^{-1}-r_i^*)}{\prod_{j=1}^{b}(z^{-1}-p_j^*)}}_{=T^*(z^{-*}),\text{ stable, anti-causal}},$$

assuming $|r_i|\langle 1$, $|p_i|\langle 1$, and $\sigma_e^2\rangle 0$.



2.2 Additive Decomposition

Let the sequence $\{f_k\}$ have a $\mathscr{Z}$-transform that exist in an annulus containing the unit circle. Then

$$F(z) = \sum_{k=-\infty}^{\infty} f_k z^{-k} = \underbrace{\sum_{k=0}^{\infty} f_k z^{-k}}_{[F(z)]_+ \text{ casual part}} + \underbrace{\sum_{k=-\infty}^{-1} f_k z^{-k}}_{[F(z)]_- \text{ strictly anti-causal part}}$$

2.3 Solving the Wiener-Hopf equation

Original problem:

$$\xrightarrow{y_k} \boxed{H^c_{\hat{x}y}(z)} \xrightarrow{\hat{x}_{k+n}}$$

$$r_{xy}(\ell+n) = \sum_{i=0}^{\infty} h_i r_{yy}(\ell - i), \quad \forall \ell \geq 0$$

New problem:

$$\xrightarrow{y_k} \boxed{1/T(z)} \xrightarrow{e_k} \boxed{H^c_{\hat{x}e}(z)} \xrightarrow{\hat{x}_{k+n}}$$

Different filter coefficients

$$\begin{aligned}
r_{xe}(\ell+n) &= \sum_{i=0}^{\infty} \bar{h}_i r_{ee}(\ell - i) = \sum_{i=0}^{\infty} \bar{h}_i \delta(\ell - i), \quad \forall \ell \geq 0 \\
\Longrightarrow \bar{h}_i &= \begin{cases} r_{xe}(i+n), & i \geq 0 \\ 0, & i < 0 \end{cases} \\
\Longrightarrow H^c_{\hat{x}e}(z) &= \left[\Phi_{xe}(z) z^n\right]_+
\end{aligned}$$

Putting it all together

$$w_k = \begin{pmatrix} e_k \\ x_k \end{pmatrix} \qquad \longrightarrow \tilde{\Phi}_{ww} = \begin{pmatrix} \tilde{\Phi}_{ee} & \tilde{\Phi}_{ex} \\ \tilde{\Phi}_{xe} & \tilde{\Phi}_{xx} \end{pmatrix}$$

$$u_k = \begin{pmatrix} y_k \\ x_k \end{pmatrix} \qquad \longrightarrow \tilde{\Phi}_{uu} = \begin{pmatrix} \tilde{\Phi}_{yy} & \tilde{\Phi}_{yx} \\ \tilde{\Phi}_{xy} & \tilde{\Phi}_{xx} \end{pmatrix}$$

Super formula:

$$\xrightarrow{u_k} \boxed{G(z)} \xrightarrow{w_k} \qquad \tilde{\Phi}_{ww} = G(z) \tilde{\Phi}_{ww} G^*(z^{-*})$$

$$G(z) = \begin{pmatrix} 1/T(z) & 0 \\ 0 & 1 \end{pmatrix} \qquad \Longrightarrow \quad \tilde{\Phi}_{xe} = \tilde{\Phi}_{xy} / T^*(z^{-*})$$

$$\Longrightarrow H^c_{\hat{x}e}(z) = \left[\frac{z^n \tilde{\Phi}_{xy}(z)}{T^*(z^{-*})}\right]_+ \qquad \Longrightarrow \quad H^c_{\hat{x}y}(z) = \frac{1}{T(z)} \cdot \left[\frac{z^n \tilde{\Phi}_{xy}(z)}{T^*(z^{-*})}\right]_+$$

Note: A factor z^{-n} is in some books added to $H^c_{\hat{x}y}(z)$. Without this factor (as given above) $\hat{x}_{k+n|k} = H^c_{\hat{x}y}(\Delta) y_k$ and with the factor $\hat{x}_{k+n|k} = \Delta^{-n} H^c_{\hat{x}y}(\Delta) y_{k+n}$. Be sure to know which convention you are adhering to, and both works just fine.