Lecture #4

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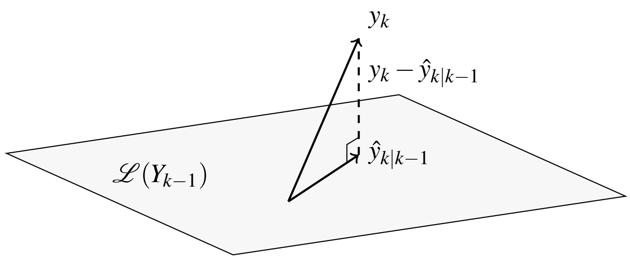
1 The Innovation Process

Let $\{y_k\}$ be a sequence of random variables, and let $\hat{y}_{k|k-1}$ be the llse of y_k given $y_0, y_1, \dots, y_{k-1}$. The "new information" brought by y_k, that could not be determined from past observations, are

$$e_k = y_k - \hat{y}_{k|k-1},$$

which is denoted the innovation.

Note that e_k is a linear function of $\{y_i\}_{i=0}^{k}$. $e_k \perp \mathscr{L}(\{y_i\}_{i=0}^{k-1})$ by the projection principle, thus e_k is a **white sequence (process)**.



To see this: Let

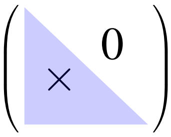
$$e_k = AY_k, \quad Y_k = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_k \end{pmatrix},$$

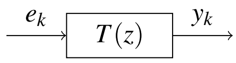
then

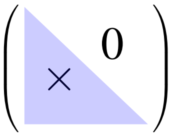
$$\begin{aligned} \mathsf{E}[e_k e_l^*] &= \mathsf{E}[e_k (AY_l^*)] = \mathsf{E}[e_k Y_l^*]A^* = \langle e_k, Y_l \rangle A^* = 0, \quad k > l \\ \mathsf{E}[e_k e_l^*] &= A\,\mathsf{E}[Y_k e_l^*] = A\langle Y_k, e_l \rangle = 0, \quad l > k \end{aligned}$$

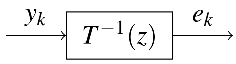
Note: In Homework 2 you showed that $E_k^T = \begin{pmatrix} e_0^T & \dots & e_k^T \end{pmatrix}$ can be obtained by a **causal** linear transformation of $Y_k^T = \begin{pmatrix} y_0^T & \dots & y_k^T \end{pmatrix}$. This transformation is casually invertible.

Matrix form | **Transfer function**

$Y_k =$  $E_k, \quad E_k = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_k \end{pmatrix}$



$E_k =$  $Y_k, \quad Y_k = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_k \end{pmatrix}$



$$R_{yy} = T R_{ee} T^*,$$

T— lower triangular, with unit diagonal

$$\Phi_{yy}(z) = \sigma_e^2 T(z) T(z^{-1}),$$
$$0 < \Phi_{yy}(z)\Big|_{z=e^{j\omega}} < \infty, \forall \omega$$

The "matrix" transformation can be obtained by, *e.g.*, Gram-Schmidt orthogonalization, *i.e.*,

$$e_k = y_k - \sum_{j=0}^{k-1} \langle y_k, e_j \rangle \|e_j\|^{-2} e_j$$

Hence, the same information is contained in Y_k and E_k, which implies that

$$\begin{aligned} \hat{x}_{l|k} &= \text{llse. of } x_l \text{ given } Y_k = \text{llse. of } x_l \text{ given } E_k \\ &= \mathrm{Proj}\big(x_l | \mathscr{L}(E_k)\big) = \big/ \text{orthogonal} \big/ = \sum_{i=0}^{k} \langle x_l, e_i \rangle \|e_i\|^{-2} e_i \\ &= \hat{x}_{l|k-1} + \underbrace{\langle x_l, e_k \rangle}_{\mathsf{E}[x_l e_k^*]} \underbrace{\|e_k^{-2}\|}_{R_{e,k}^{-1}} e_k \end{aligned}$$

Summary

$$\hat{x}_{l|k} = \hat{x}_{l|k-1} + \langle x_l, e_k \rangle \|e_k\|^{-2} e_k \quad \text{(Recursive update of llse.)}$$
$$e_k = y_k - \hat{y}_{k|k-1} = y_k - \sum_{j=0}^{k-1} \langle y_k, e_j \rangle \|e_j\|^{-2} e_j \quad \text{(Innovations sequence from observations)}$$

See how the Wiener-Hopf equations were solved in Lecture 2 using the whitening filter.

Computational issues

- Generally: $\mathscr{O}(k^3)$ operation to calculate T (factorize R_{yy}) or equivalently the innovations.
- If $\{y_k\}$ stationary process $\rightarrow \mathscr{O}(k^2)$ operations.
- State-space model, where $\dim(x_k) = n \ll k \rightarrow \mathscr{O}(kn^3)$ operations.

Kalman Filter — an innovation approach

State-space model ($k \geq 0$):

$$
\begin{aligned}
x_{k+1} &= F_k x_k + G_k w_k \\
y_k &= H_k x_k + v_k \\
\mathsf{E}\left[\begin{pmatrix} w_k \\ v_k \\ x_0 \end{pmatrix}\begin{pmatrix} w_l \\ v_l \\ x_0 \\ 1 \end{pmatrix}^*\right] &= \begin{pmatrix} Q_k\delta_{l-k} & S_k\delta_{l-k} & 0 & 0 \\ S_k^*\delta_{l-k} & R_k\delta_{l-k} & 0 & 0 \\ 0 & 0 & \Pi_0 & 0. \end{pmatrix}
\end{aligned}
$$

(The last column implies the involved stochastic variables are zero-mean.) Let $e_k = y_k - \hat{y}_{k|k-1}$, where

$$
\hat{y}_{k|k-1} = \mathrm{Proj}(y_k | Y_{k-1}) = \mathrm{Proj}(H_k x_k + v_k | Y_{k-1}) = \Big/ v_k \perp Y_{k-1} \Big/ = H_k \hat{x}_{k|k-1}
$$

Thus, finding the innovations is equivalent to find the one-step predictor of the state x_k. Since $\{e_k\}$ is a white sequence, then

$$
\begin{aligned}
\hat{x}_{k+1|k} &= \sum_{i=0}^{k} \langle x_{k+1}, e_i \rangle \|e_i\|^{-2} e_i \\
&= \hat{x}_{k+1|k-1} + \langle x_{k+1}, e_k \rangle \|e_k\|^{-2} e_k \\
&= \hat{x}_{k+1|k-1} + K_{p,k} e_k, \qquad\qquad K_{p,k} = \langle x_{k+1}, e_k \rangle \|e_k\|^{-2} = \mathsf{E}\left[x_{k+1} e_k^*\right]
\end{aligned}
$$

Note, we would like an expression in terms of $\hat{x}_{k|k-1}$ instead of $\hat{x}_{k+1|k-1}$.

$$
\begin{aligned}
\hat{x}_{k+1|k-1} &= \mathrm{Proj}(x_{k+1} | Y_{k-1}) = \mathrm{Proj}(F x_k + G_k w_k | Y_{k-1}) \\
&= F_k \,\mathrm{Proj}(x_k | Y_{k-1}) = F_k \hat{x}_{k|k-1}
\end{aligned}
$$

Bringing it all to together we get the recursion

$$
\hat{x}_{k+1|k} = F_k \hat{x}_{k|k-1} + K_{p,k}(y_k - H_k \hat{x}_{k|k-1}) = (F_k - K_{p,k} H_k) \hat{x}_{k|k-1} + K_{p,k} y_k .
$$

What remains is to find a recursive way to calculate the Kalman predictive gain $K_{p,k}$.

Let $\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}$ and $P_{k|k-1} = \mathsf{E}\left[\tilde{x}_{k|k-1} \tilde{x}_{k|k-1}^*\right]$ then

$$
\begin{aligned}
e_k &= y_k - H_k \hat{x}_{k|k-1} = H_k \tilde{x}_{k|k-1} + v_k \\
R_{e,k} &= \mathsf{E}\left[e_k e_k^*\right] = \Big/ v_k \perp \tilde{x}_{k|k-1} \Big/ = H_k P_{k|k-1} H_k^* + R_k
\end{aligned}
$$

Further we have that

$$
\mathsf{E}\left[x_{k+1} e_k^*\right] = F_k \,\mathsf{E}\left[x_k e_k^*\right] + G_k \,\mathsf{E}\left[w_k e_k^*\right]
$$

where

$$
\begin{aligned}
\mathsf{E}\left[x_k e_k^*\right] &= \mathsf{E}\left[x_k (H_k \tilde{x}_{k|k-1} + v_k)^*\right] \\
&= \mathsf{E}\left[x_k \tilde{x}_{k|k-1}^*\right] H_k^* + \mathsf{E}\left[x_k v_k^*\right] = \Big/ \hat{x}_{k|k-1} \perp \tilde{x}_{k|k-1} \Rightarrow \mathsf{E}\left[\hat{x}_{k|k-1} \tilde{x}_{k|k-1}^*\right] = 0 \Big/ \\
&= \mathsf{E}\left[(x_k - \hat{x}_{k|k-1}) \tilde{x}_{k|k-1}^*\right] H_k^* = P_{k|k-1} H_k^* \\
\mathsf{E}\left[w_k e_k^*\right] &= \mathsf{E}\left[w_k (H_k \tilde{x}_{k|k-1} + v_k)^*\right] = \mathsf{E}\left[w_k v_k^*\right] = S_k
\end{aligned}
$$

yielding

$$K_{p,k} = (F_k P_{k|k-1} H_k^* + G_k S_k)(H_k P_{k|k-1} H_k^* + R_k)^{-1}$$

Now we need a recursion for $P_{k|k-1}$.

$$\begin{aligned}
\tilde{x}_{k+1|k} &= x_{k+1} - \hat{x}_{k+1|k} = F_k x_k + G_k w_k - (F_k \hat{x}_{k|k-1} + K_{p,k} e_k) \\
&= F_k x_k + G_k w_k - F_k \hat{x}_{k|k-1} - K_{p,k}(H_k \tilde{x}_{k|k-1} + v_k) \\
&= (F_k - K_{p,k} H_k)\tilde{x}_{k|k-1} + \begin{pmatrix} G_k & K_{p,k} \end{pmatrix} \begin{pmatrix} w_k \\ -v_k \end{pmatrix}
\end{aligned}$$

yielding ($\tilde{x}_{k|k-1} \perp w_k, v_k$)

$$\begin{aligned}
P_{k+1|k} &= (F_k - K_{p,k} H_k) P_{k|k-1} (F_k - K_{p,k} H_k)^* + \begin{pmatrix} G_k & K_{p,k} \end{pmatrix} \begin{pmatrix} Q_k & -S_k \\ -S_k^* & R_k \end{pmatrix} \begin{pmatrix} G_k & K_{p,k} \end{pmatrix}^* \\
&= \Big/ \ldots \Big/ = F_k P_{k|k-1} F_k^* + G_k Q_k G_k^* - K_{p,k} R_{e,k} K_{p,k}^*
\end{aligned}$$

This is the *discrete time algebraic Riccati equation* (DARE)

Kalman Filter (prediction form)

$P_0 = \Pi_0, \quad \hat{x}_{0\|-1} = 0$ (assuming zero-mean)	Initial values
$e_k = y_k - H_k \hat{x}_{k\|k-1}$	Innovation
$R_{e,k} = H_k P_{k\|k-1} H_k^* + R_k$	Innovation covariance
$K_{p,k} = (F_k P_{k\|k-1} H_k^* + G_k S_k) R_{e,k}^{-1}$	Kalman prediction gain
$\hat{x}_{k+1\|k} = F_k \hat{x}_{k\|k-1} + K_{p,k} e_k$	Prediction
$P_{k+1\|k} = F_k P_{k\|k-1} F_k^* + G_k Q_k G_k^* - K_{p,k} R_{e,k} K_{p,k}^*$	State covariance

$K_{p,k}$ tells us how much we should adjust our estimate $\hat{x}_{k|k-1}$ given observations y_k:

- $K_{p,k}$ — small: Trust the model ($\|Q\|/\|R\|$ small)
- $K_{p.k}$ — large: Trust the measurements/observations ($\|Q\|/\|R\|$ large)