Lecture #5

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Assumptions (today's lecture):

- Time invariant system, *i.e.*, $F_k = F$, $G_k = G$, and $H_k = H$.
- $\{v_k\}$ and $\{w_k\}$ stationary zero-mean process, *i.e.*, $R_k = R$, $Q_k = Q$, and $S_k = S$.
- The system is stable, *i.e.*, $|\lambda_i(F)| < 1, \forall i$.

Time-Invariant System

$$x_{k+1} = Fx_k + Gw_k, \qquad \mathsf{E}[x_0] = 0$$
$$y_k = Hx_k + v_k, \qquad \mathsf{E}[x_0 x_0^*] = \Pi_0$$

yielding

$$\Pi_{k+1} = F\Pi_k F^* + GQG^*$$

Hence, even if $\{v_k\}$ and $\{w_k\}$ are stationary processes, $\{x_k\}$ $\{y_k\}$ will generally not be stationary. They are stationary if $\Pi_{k+1} = \Pi_k = \bar{\Pi}$ and $\bar{\Pi} \succ 0$ (positive definite), and given by the Lyaponov equation

$$\bar{\Pi} = F\bar{\Pi}F^* + GQG^*.$$

Generally, several solutions exist. If $\{F, G(Q^{\frac{1}{2}})^*\}$ controllable and $|\lambda(F)| < 1, \forall i$, then the Lyaponov equation has a unique solution $\Pi_s \succ 0$. Hence then if

1. $\Pi_0 = \bar{\Pi}_s$, $\{x_k\}$ and $\{y_i\}$ are stationary processes.
2. $\Pi_0 \neq \bar{\Pi}_s$, $\{x_k\}$ and $\{y_k\}$ become stationary processes as $k \to +\infty$.

Proof of Unique Positive Definite Solution

Note: $\operatorname{vec}(AXB) = (B^* \otimes A)\operatorname{vec}(X)$ ($\otimes$ represents the Kronecker product), now

$$\bar{\Pi} = F\bar{\Pi}F^* + GQG^* \quad \Leftrightarrow \quad (I - F \otimes F)\operatorname{vec}(\bar{\Pi}) = \operatorname{vec}(GQG^*)$$

Linear system of equations has a unique solution if $\lambda_i(I - F \otimes F) \neq 0, \forall i$.

$$\lambda_i(I - F \otimes F) \neq 0, \forall i \quad \Leftrightarrow \quad \lambda_i(F \otimes F) \neq 1, \forall i \quad \Leftrightarrow \quad \lambda_i(F)\lambda_j(F) \neq 1, \forall i, j$$

If F is stable, the $\lambda_i(F) < 1, \forall i \Rightarrow \lambda_i(F)\lambda_j(F) < 1, \forall i, j$ and we have a unique solution. Further, if F is stable, then the series

$$\sum_{i=0}^{\infty} F^i G Q G^* (F^*)^i \to \tilde{\Pi} \quad \text{(converges to } \tilde{\Pi}\text{)}.$$

Next, note that that

$$\tilde{\Pi} - F\tilde{\Pi}F^* = \sum_{i=0}^{\infty} F^i G Q G^* (F^*)^i - \sum_{i=1}^{\infty} F^i G Q G^* (F^*)^i = GQG^*$$

Hence, $\tilde{\Pi}$ satisfies the same Lyaponov equation as $\bar{\Pi}$, but the uniqueness of the solution implies that $\tilde{\Pi} = \bar{\Pi}$. Next,

$$\begin{aligned} \tilde{\Pi} &= \sum_{i=0}^{\infty} F^i G Q G^* (F^*)^i \geq \sum_{i=0}^{n-1} F^i G Q G^* (F^*)^i \\ &= \underbrace{\begin{pmatrix} GQ^{\frac{*}{2}} & FGQ^{\frac{*}{2}} & \dots & F^{n-1}GQ^{\frac{*}{2}} \end{pmatrix}}_{M} \underbrace{(\star)^*}_{M^*} \end{aligned}$$

where $n = \dim(x_k)$, then $\tilde{\Pi} \succ 0$ (positive definite) if $\text{rank}(M) = n$.

$$\text{rank}(M) = \text{rank}\Big(\begin{pmatrix} GQ^{\frac{*}{2}} & FGQ^{\frac{*}{2}} & \dots & F^{n-1}GQ^{\frac{*}{2}} \end{pmatrix}\Big) = n \Leftrightarrow \text{Controllable } \{F, GQ^{\frac{*}{2}}\}$$

Spectra and Auto Correlation Function of $\{x\}$ and $\{y\}$

Recall: If $\{a_k\}$ is a zero-mean stationary process, yielding

$$\begin{aligned} \text{acf:} \quad & \mathsf{E}[a_l a_k^*] = r_a(l-k) \\ \text{Spectrum:} \quad & \Phi_{aa}(z) = \sum_{k=-\infty}^{\infty} r_a(k) z^{-k}, \quad \rho < |z| < \rho^{-1},\ 0 < \rho < 1 \\ \text{Superformula:} \quad & H(z) : \{a_k\} \to \{b_k\} \Rightarrow \Phi_{bb}(z) = H(z)\Phi_{aa}(z)H^*(z^{-*}) \end{aligned}$$

Acf of $\{x_k\}$:

$$\mathsf{E}[x_{k+l}x_k^*] = \begin{cases} F^l \bar{\Pi}, & l \geq 0 \\ \bar{\Phi}(F^*)^{-l}, & l < 0 \end{cases}$$

Spectrum of $\{x_k\}$:

$$\begin{aligned} x_{k+1} = Fx_k + Gw_k &\to H_{xw}(z) = (zI - F)^{-1} G \\ \Rightarrow \Phi_{xx}(z) &= (zI - F)^{-1} G Q G^* (z^{-1} I - F^*)^{-1} \end{aligned}$$

Spectrum of $\{y_k\}$:

$$\begin{aligned} x_{k+1} &= Fx_k + Gw_k \\ y_k &= Hx_k + v_k \end{aligned}$$

Let $u_k = \begin{pmatrix} w_k^* & v_k^* \end{pmatrix}^*$, then $H_{yu}(z) = \begin{pmatrix} H(zI-F)^{-1}G & I \end{pmatrix}$, yielding

$$\Phi_{yy}(z) = \begin{pmatrix} H(zI-F)^{-1}G & I \end{pmatrix} \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \begin{pmatrix} G^*(z^{-*}I - F^*)^{-1}H^* \\ I \end{pmatrix}$$

Case $S = 0$:

$$\Phi_{yy}(z) = R + H\Phi_{xx}(z)H^*$$

Time Invariant (Stationary) Kalman Filter

KF recursion:

$$
\begin{aligned}
e_k &= y_k - H\hat{x}_{k|k-1} \\
R_{e,k} &= HP_{k|k-1}H^* + R \\
K_{p,k} &= (FP_{k|k-1}H^* + GS)R_{e,k}^{-1} \\
\hat{x}_{k+1|k} &= F\hat{x}_{k|k-1} + K_{p,k}e_k \\
P_{k+1|k} &= FP_{k|k-1}F^* + GQG^* - K_{p,k}R_{e,k}K_{p,k}^* \qquad (\star)
\end{aligned}
$$

Assume that $(\star)$ converge so that $\lim_{k\to\infty} P_{k|k-1} = \bar{P} \Rightarrow K_{p,k} \to \bar{K}_p$ and the Kalman filter becomes time invariant with the transfer function

$$H_{\hat{x}y}(z) = (zI - F + \bar{K}_pH)^{-1}\bar{K}_p$$

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k-1} + K_{p,k}(y_k - H\hat{x}_{k|k-1}) = (F - K_{p,k}H)\hat{x}_{k|k-1} + \bar{K}_{p,k}y_k$$

This filter is a stable filter if $|\lambda_i(F - \bar{K}_pH)| < 1, \forall i$. **Note:** This **does not** require F to represent a stable system, $|\lambda_i(F)| < 1, \forall i$!!!

When does $(\star)$ converge?

If $|\lambda_i(F)| < 1, \forall i$ (stable system) or if $\{F, H\}$ is detectable and $\{F^s, GQ^{\frac{s}{2}}\}$ controllable on the unit circle ($F^s = F - GSR^{-1}H$, $Q^s = Q - SR^{-1}S^*$), then the DARE

$$\bar{P} = F\bar{P}F^* + GQG^* - \bar{K}_pR_e^{-1}\bar{K}_p$$

has a unique solution for which $R_e \succ 0$ and $|\lambda_i(F - K_pH)| < 1, \forall i$. Further, if $\{F^s, GQ^{\frac{s}{2}}\}$ is stabilazible, then $\bar{P} \succeq 0$.

Simplified:

- If $|\lambda_i(F)| < 1, \forall i$ you are generally fine.
- If there exist $|\lambda_i(F)| \geq 1$, then unstable modes must be observable and excited by process noise.

Observability: Consider the noise free system

$$
\begin{aligned}
x_{k+1} &= Fx_k \\
y_k &= Hx_k
\end{aligned}
$$

then x_l can only be uniquely determined from $y_l, y_{l+1}, \ldots$ if

$$\text{rank}\left(\begin{pmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{pmatrix}\right) = n \quad \text{(full rank)},$$

then the system is observable.

Innovation Model and Spectral Factorization of $\Phi_{yy}(z)$

If we have a stable system, then as $P_{k|k-1} \to \bar{P}$, we can describe y_k (as $k \to \infty$) using the time invariant innovation model

$$\begin{aligned} \hat{x}_{k+1|k} &= F\hat{x}_{k|k-1} + \bar{K}_p e_k \\ y_k &= H\hat{x}_{k|k-1} + e_k \end{aligned}$$

- Transfer function $T(z) : \{e_k\} \to \{y_k\}$, $T(z) = I + H(zI - F)^{-1}\bar{K}_p$.
- $\{e_k\}$ white noise with covariance $R_e = HPH^* + R$, which yields $\Phi_{yy}(z) = T(z)R_eT^*(z^{-*})$
- $T(z)$ stable since $|\lambda_i(F)| < 1, \forall i$.
- $T^{-1}(z) = \big/\text{matrix inversion lemma}\big/ = I - H(zI - (F - \bar{K}_pH)^{-1})^{-1}\bar{K}_p$ which is stable since $|\lambda_i(F - \bar{K}_pH)| < 1, \forall i$.