Lecture #6

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Smoothing

In many applications an delayed estimate might be tolerable or off-line processing is interesting. This way, more data can be used in the estimate, providing

- higher accuracy
- more complex estimators.

Different Types of Smoothers

Assume:



Applications

- Fixed point smoothing: Initial condition estimation.
- Fixed lag smoothing: Online processing with delay.
- Fixed interval smoothing: Offline processing.

Fixed Point Smoothing (see ex 4 in HW #3)

Goal: Find $\hat{x}_{j|k}$ for all k > j, *j*-fixed

Trick: Introduce the augmented state vector $(x_k^* (x_k^a)^*)^*$ and the augmented state-space model

$$\begin{pmatrix} x_{k+1} \\ x_{k+1}^a \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_k \\ x_k^a \end{pmatrix} + \begin{pmatrix} G_k \\ 0 \end{pmatrix} w_k$$
$$y_k = \begin{pmatrix} H_k & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_k^a \end{pmatrix} + v_k.$$

Note, the dynamic transition of the augmented part of the state $x_k^a \equiv x_j$ is constant. Apply the Kalman filter recursion

Gain:

$$\begin{pmatrix}
K_{p,k} \\
K_{p,k}^{a}
\end{pmatrix} = \begin{pmatrix}
F_{k} & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
P_{k|k-1} & (P_{k|k-1}^{a})^{*} \\
P_{k|k-1}^{a} & P_{k|k-1}^{aa}
\end{pmatrix} \begin{pmatrix}
H_{k}^{*} \\
0
\end{pmatrix} (H_{k}P_{k|k-1}H_{k}^{*} + R_{k})^{-1},$$

where $P_{k|k-1}^{aa} = P_{k|k}^{aa} = \mathsf{E}\left[(x_j - \hat{x}_{j|k})(x_j - \hat{x}_{j|k})^*\right]$, yielding

$$K_{p,k} = F_k P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}$$

$$K_{p,k}^a = F_k P_{k|k-1}^a H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}$$

Covariance:

$$\begin{pmatrix} P_{k+1|k} & (P_{k+1|k}^{a})^{*} \\ P_{k+1|k}^{a} & P_{k+1|k}^{aa} \end{pmatrix} = \begin{pmatrix} F_{k} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^{a})^{*} \\ P_{k|k-1}^{a} & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} F_{k}^{*} & 0 \\ 0 & I \end{pmatrix}$$
$$- \begin{pmatrix} K_{p,k} \\ K_{p,k}^{a} \end{pmatrix} (H_{k} & 0) \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^{a})^{*} \\ P_{k|k-1}^{a} & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} F_{k}^{*} & 0 \\ 0 & I \end{pmatrix}$$
$$+ \begin{pmatrix} G_{k} \\ 0 \end{pmatrix} Q_{k} (G_{k}^{*} & 0)$$

Remark: $K(HPH^* + R)K^* = K(HPH^* + R)(HPH^* + R)^{-1}HPF^* = KHPF^*$

$$\begin{split} P_{k+1|k} &= F_k P_{k|k-1} F_k^* - K_{p,k} H_k P_{k|k-1} F_k^* + G_k Q G_k^* \quad \text{standard update} \\ P_{k+1|k}^a &= P_{k|k-1}^a F_k^* - K_{p,k}^a H_k P_{k|k-1} F_k^* \\ P_{k+1|k}^{aa} &= P_{k|k-1}^a - K_{p,k}^a H_k P_{k|k-1}^a \end{split}$$

Estimate:

$$\begin{aligned} \hat{x}_{j|k} &= \hat{x}_{j|k-1} + K_{p,k} e_k \\ e_k &= y_k - \begin{pmatrix} H_k & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_{k|k-1} \\ \hat{x}_{k|k-1}^a \end{pmatrix} = y_k - H \hat{x}_{k|k-1} \end{aligned}$$

Summary: Standard Kalman filter recursions

$$\begin{aligned} \hat{x}_{j|k} &= \hat{x}_{j|k-1} + K^{a}_{p,k}(y_{k} - H_{k}\hat{x}_{k|k-1}) \\ K^{a}_{p,k} &= P^{a}_{k|k-1}H^{*}_{k}R^{-1}_{e,k} \\ P^{a}_{k+1|k} &= P^{a}_{k|k-1}F^{*}_{k} - K^{a}_{p,k}H_{k}P_{k|k-1}F^{*}_{k} \\ P^{aa}_{k+1|k} &= P^{aa}_{k|k-1} - K_{p,k}H_{k}P^{a}_{k|k-1} \end{aligned}$$

Initial condition: $\hat{x}_{j|j-1}^a = \hat{x}_{j|j-1}$ and $P_{j|j-1}^a = P_{j|j-1}$

Remark:

- Most improvements are obtained after 2–3 time constants of the system.
- Time varying even for time invariant systems.

Fixed Lag Smoother

Goal: Find $\hat{x}_{k-n|k}$ and $P_{k-n|k}$, *n*-fixed, *k*-growing

Trick: Introduce the state-space model

$$\begin{pmatrix} x_{k+1} \\ x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \\ \vdots \\ x_{k+1}^{(n+1)} \end{pmatrix} = \begin{pmatrix} F_k & 0 & \dots & 0 \\ I & 0 & \ddots & 0 \\ 0 & I & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & I & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_k^{(1)} \\ x_k^{(2)} \\ \vdots \\ x_k^{(n+1)} \end{pmatrix} + \begin{pmatrix} G_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} w_k$$

$$y_k = \begin{pmatrix} H_k & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_k^{(1)} \\ x_k^{(2)} \\ \vdots \\ x_k^{(n+1)} \end{pmatrix} + v_k$$

Note that: $x_{k+1}^{(1)} = x_k, x_{k+1}^{(2)} = x_{k-1}, \dots, x_{k+1}^{(n+1)} = x_{k-n}$, yielding $\hat{x}_{k+1|k}^{(i)} = x_{k-i+1|k}, i = 1, \dots, n+1$.

Hence: The one-step predictor of the augmented state-space model will give the l.l.s.e. of x_{k-i} given $\{y_j\}_{j=0}^k$ for i = 0, ..., n.

Fixed Lag Smoothing Equations: Standard Kalman filter recursions for the original state-space model. 1: $\hat{x}_{0|-1} = \hat{x}_{-i|-1} = 0$ 2: $P_{k|k-1}^{(0)} = P_{k|k-1}$ 3: for i=1,..., N do 4: $\hat{x}_{k+1|k}^{(i+1)} = \hat{x}_{k|k-1}^{(i)} + K_{p,k}^{(i+1)} e_k$ 5: $K_{p,k}^{(i+1)} = P_{k|k-1}^{(i)} H_k^* R_{e,k}^{-1}$ 6: $P_{k+1|k}^{(i+1)} = P_{k|k-1}^{(i)} (F_k - K_{p,k} H_k)^*$ 7: $P_{k-i|k} = P_{k-i|k-1} - P_{k|k-1}^{(i)} H_k^* (K_{p,k}^{(i)})^*$ 8: end for

Remark:

- Time invariant for time invariant state-space model.
- If *n* corresponds to 2–3 time constants almost all info has been included in the estimate $\hat{x}_{k-n|k}$, hence a fixed interval smoother gives marginal improvement.

Fixed Interval Smoother

See Bryson-Frazier and Rausch-Tung-Stribel formulas in the book.

Information Filter

If the initial state is completely unknown, then $P_0 \to \infty$, which may cause numerical problems. However, $P_0^{-1} = 0$ is quite natural. Propagating P_k^{-1} instead of P_k is referred to the information formulation of the Kalman filter.

Note: for Gaussian linear state-space systems, then P_k^{-1} is equivalent to the Fisher information matrix.

Derivation of the Information Filter

Assume:

- $S_k = 0$ (for simplicity)
- F_k non-singular (for simplicity)

Recall:

- $(A + BCD)^{-1} = A^{-1} A^{-1}B(C^{-1} + DA^{-1}B)DA^{-1}$
- Kalman filter gain: $K_{f,k} = P_{k|k-1}H_k^*R_{e,k}^{-1} = P_{k|k}H_k^*R_k^{-1}$

Measurement update of P_k^{-1}

$$\begin{aligned} P_{k|k} &= P_{k|k-1} - P_{k|k-1} H_k^* (H_k P_{k|k} H_k^* + R_k)^{-1} H_k P_{k|k-1} \\ &= (P_{k|k-1} + H_k^* R_k^{-1} H_k)^{-1} \\ P_{k|k}^{-1} &= P_{k|k-1}^{-1} + \underbrace{H_k^* R_k^{-1} H_k}_{\text{Info. added by the meas.}} \end{aligned}$$

Time update of P_k^{-1}

$$\begin{split} P_{k+1|k} &= F_k P_{k|k} F_k^* + G_k Q_k G_k^* \\ P_{k+1|k}^{-1} &= (F_k P_{k|k} F_k^* + G_k Q_k G_k^*)^{-1} = \{A_k = F_k^{-*} P_{k|k}^{-1} F_k^{-1}\} \\ &= (A_k^{-1} + G_k Q_k G_k^*)^{-1} \\ &= A_k - \underbrace{A_k G_k (G_k^* A_k G_k + Q_k^{-1})^{-1} G_k^* A_k}_{\text{reduction in information when predicting}} \\ &= (I - A_k G_k (G_k^* A_k G_k + Q_k^{-1})^{-1} G_k^*) A_k \\ &= (I - B_k G_k^*) A_k, \end{split}$$

where $B_k = A_k G_k (G_k^* A_k G_k + Q_k^{-1})^{-1}$.

State Recursions

Definition: $\hat{a}_{k|k-1} = P_{k|k-1}^{-1} \hat{x}_{k|k-1}$ and $\hat{a}_{k|k} = P_{k|k}^{-1} \hat{x}_{k|k}$

Measurement update:

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + P_{k|k} H_k^* R_k^{-1} (y_k - H_k \hat{x}_{k|k-1}) \\ P_{k|k}^{-1} \hat{x}_{k|k} &= P_{k|k}^{-1} \hat{x}_{k|k-1} + H_k^* R_k^{-1} y_k - H_k^* R_k^{-1} H_k \hat{x}_{k|k-1} \\ &= P_{k|k-1}^{-1} \hat{x}_{k|k-1} + H_k^* R_k^{-1} y_k \\ \hat{a}_k &= \hat{a}_{k|k-1} + H_k^* R_k^{-1} y_k \end{aligned}$$

Time update:

$$\begin{split} \hat{x}_{k+1|k} &= F_k \hat{x}_{k|k} \\ \hat{a}_{k+1|k} &= P_{k+1|k}^{-1} \hat{x}_{k+1|k} = P_{k+1|k}^{-1} F_k \hat{x}_{k|k} = (I - B_k G_k^*) A_k F_k \hat{x}_{k|k} \\ &= (I - B_k G_k^*) F_k^{-*} \hat{a}_{k|k} \end{split}$$

Summary:

Measurement update:

$$\hat{a}_{k|k} = \hat{a}_{k|k-1} + H_k R_k^{-1} y_k$$

 $P_{k|k}^{-1} = P_{k|k-1}^{-1} + H_k^* R_k^{-1} H_k$

Time update:

$$\hat{a}_{k+1|k} = (I - B_k G_k^*) F_k^{-*} \hat{a}_{k|k}$$

 $P_{k+1|k} = (I - B_k G_k^*) A_k$

where

$$A_k = F_k^{-*} P_{k|k} F_k^{-1}$$
$$B_k = A_k G_k (G_k^* A_k G_k + Q_k)^{-1}$$