

# Lecture #6

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## Smoothing

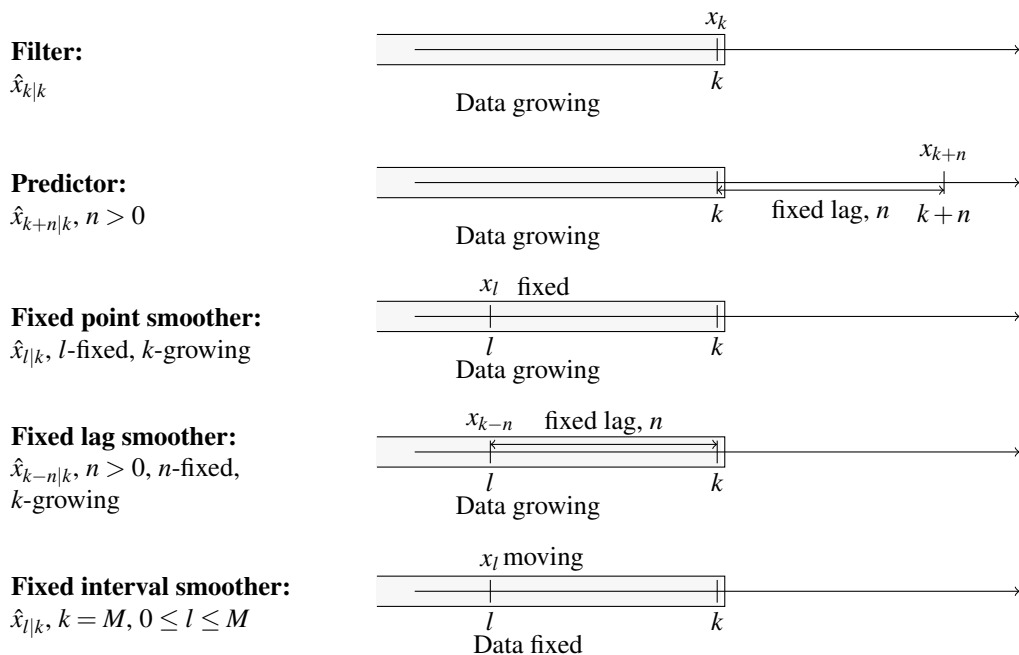
In many applications an delayed estimate might be tolerable or off-line processing is interesting. This way, more data can be used in the estimate, providing

- higher accuracy
- more complex estimators.

## Different Types of Smoothers

Assume:

$$\hat{x}_{l|k} = \text{llse of } x_l \text{ given } \{y_0, y_1, \dots, y_k\}$$



## Applications

- Fixed point smoothing: Initial condition estimation.
- Fixed lag smoothing: Online processing with delay.
- Fixed interval smoothing: Offline processing.

## Fixed Point Smoothing (see ex 4 in HW #3)

**Goal:** Find  $\hat{x}_{j|k}$  for all  $k > j$ ,  $j$ -fixed

**Trick:** Introduce the augmented state vector  $(x_k^* \quad (x_k^a)^*)^*$  and the augmented state-space model

$$\begin{pmatrix} x_{k+1} \\ x_{k+1}^a \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_k \\ x_k^a \end{pmatrix} + \begin{pmatrix} G_k \\ 0 \end{pmatrix} w_k$$

$$y_k = (H_k \quad 0) \begin{pmatrix} x_k \\ x_k^a \end{pmatrix} + v_k.$$

Note, the dynamic transition of the augmented part of the state  $x_k^a \equiv x_j$  is constant.

Apply the Kalman filter recursion

**Gain:**

$$\begin{pmatrix} K_{p,k} \\ K_{p,k}^a \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^a)^* \\ P_{k|k-1}^a & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} H_k^* \\ 0 \end{pmatrix} (H_k P_{k|k-1} H_k^* + R_k)^{-1},$$

where  $P_{k|k-1}^{aa} = P_{k|k}^{aa} = E[(x_j - \hat{x}_{j|k})(x_j - \hat{x}_{j|k})^*]$ , yielding

$$K_{p,k} = F_k P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}$$

$$K_{p,k}^a = F_k P_{k|k-1}^a H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}$$

**Covariance:**

$$\begin{pmatrix} P_{k+1|k} & (P_{k+1|k}^a)^* \\ P_{k+1|k}^a & P_{k+1|k}^{aa} \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^a)^* \\ P_{k|k-1}^a & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} F_k^* & 0 \\ 0 & I \end{pmatrix}$$

$$- \begin{pmatrix} K_{p,k} \\ K_{p,k}^a \end{pmatrix} (H_k \quad 0) \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^a)^* \\ P_{k|k-1}^a & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} F_k^* & 0 \\ 0 & I \end{pmatrix}$$

$$+ \begin{pmatrix} G_k \\ 0 \end{pmatrix} Q_k (G_k^* \quad 0)$$

**Remark:**  $K(HPH^* + R)K^* = K(HPH^* + R)(HPH^* + R)^{-1}HPF^* = KHPF^*$

$$P_{k+1|k} = F_k P_{k|k-1} F_k^* - K_{p,k} H_k P_{k|k-1} F_k^* + G_k Q G_k^* \quad \text{standard update}$$

$$P_{k+1|k}^a = P_{k|k-1}^a F_k^* - K_{p,k}^a H_k P_{k|k-1} F_k^*$$

$$P_{k+1|k}^{aa} = P_{k|k-1}^{aa} - K_{p,k}^a H_k P_{k|k-1}^a$$

**Estimate:**

$$\hat{x}_{j|k} = \hat{x}_{j|k-1} + K_{p,k} e_k$$

$$e_k = y_k - \begin{pmatrix} H_k & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_{k|k-1} \\ \hat{x}_{k|k-1}^a \end{pmatrix} = y_k - H \hat{x}_{k|k-1}$$

**Summary:** Standard Kalman filter recursions

$$\hat{x}_{j|k} = \hat{x}_{j|k-1} + K_{p,k}^a (y_k - H_k \hat{x}_{k|k-1})$$

$$K_{p,k}^a = P_{k|k-1}^a H_k^* R_{e,k}^{-1}$$

$$P_{k+1|k}^a = P_{k|k-1}^a F_k^* - K_{p,k}^a H_k P_{k|k-1} F_k^*$$

$$P_{k+1|k}^{aa} = P_{k|k-1}^{aa} - K_{p,k}^a H_k P_{k|k-1}^a$$

Initial condition:  $\hat{x}_{j|j-1}^a = \hat{x}_{j|j-1}$  and  $P_{j|j-1}^a = P_{j|j-1}$

**Remark:**

- Most improvements are obtained after 2–3 time constants of the system.
- Time varying even for time invariant systems.

## Fixed Lag Smoother

**Goal:** Find  $\hat{x}_{k-n|k}$  and  $P_{k-n|k}$ ,  $n$ -fixed,  $k$ -growing

**Trick:** Introduce the state-space model

$$\begin{pmatrix} x_{k+1} \\ x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \\ \vdots \\ x_{k+1}^{(n+1)} \end{pmatrix} = \begin{pmatrix} F_k & 0 & \dots & 0 \\ I & 0 & \ddots & \\ 0 & I & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & & I & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_k^{(1)} \\ x_k^{(2)} \\ \vdots \\ x_k^{(n+1)} \end{pmatrix} + \begin{pmatrix} G_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} w_k$$

$$y_k = \begin{pmatrix} H_k & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_k^{(1)} \\ x_k^{(2)} \\ \vdots \\ x_k^{(n+1)} \end{pmatrix} + v_k$$

**Note that:**  $x_{k+1}^{(1)} = x_k, x_{k+1}^{(2)} = x_{k-1}, \dots, x_{k+1}^{(n+1)} = x_{k-n}$ , yielding  $\hat{x}_{k+1|k}^{(i)} = x_{k-i+1|k}, i = 1, \dots, n+1$ .

**Hence:** The one-step predictor of the augmented state-space model will give the l.l.s.e. of  $x_{k-i}$  given  $\{y_j\}_{j=0}^k$  for  $i = 0, \dots, n$ .

### Fixed Lag Smoothing Equations:

Standard Kalman filter recursions for the original state-space model.

- 1:  $\hat{x}_{0|-1} = \hat{x}_{-i|-1} = 0$
- 2:  $P_{k|k-1}^{(0)} = P_{k|k-1}$
- 3: **for**  $i=1, \dots, N$  **do**
- 4:  $\hat{x}_{k+1|k}^{(i+1)} = \hat{x}_{k|k-1}^{(i)} + K_{p,k}^{(i+1)} e_k$
- 5:  $K_{p,k}^{(i+1)} = P_{k|k-1}^{(i)} H_k^* R_{e,k}^{-1}$
- 6:  $P_{k+1|k}^{(i+1)} = P_{k|k-1}^{(i)} (F_k - K_{p,k} H_k)^*$
- 7:  $P_{k-i|k} = P_{k-i|k-1} - P_{k|k-1}^{(i)} H_k^* (K_{p,k}^{(i)})^*$
- 8: **end for**

#### Remark:

- Time invariant for time invariant state-space model.
- If  $n$  corresponds to 2–3 time constants almost all info has been included in the estimate  $\hat{x}_{k-n|k}$ , hence a fixed interval smoother gives marginal improvement.

### Fixed Interval Smoother

See Bryson-Frazier and Rausch-Tung-Stribel formulas in the book.

### Information Filter

If the initial state is completely unknown, then  $P_0 \rightarrow \infty$ , which may cause numerical problems. However,  $P_0^{-1} = 0$  is quite natural. Propagating  $P_k^{-1}$  instead of  $P_k$  is referred to the information formulation of the Kalman filter.

**Note:** for Gaussian linear state-space systems, then  $P_k^{-1}$  is equivalent to the Fisher information matrix.

### Derivation of the Information Filter

#### Assume:

- $S_k = 0$  (for simplicity)
- $F_k$  non-singular (for simplicity)

#### Recall:

- $(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)DA^{-1}$
- Kalman filter gain:  $K_{f,k} = P_{k|k-1} H_k^* R_{e,k}^{-1} = P_{k|k} H_k^* R_k^{-1}$

### Measurement update of $P_k^{-1}$

$$\begin{aligned}
 P_{k|k} &= P_{k|k-1} - P_{k|k-1} H_k^* (H_k P_{k|k} H_k^* + R_k)^{-1} H_k P_{k|k-1} \\
 &= (P_{k|k-1} + H_k^* R_k^{-1} H_k)^{-1} \\
 P_{k|k}^{-1} &= P_{k|k-1}^{-1} + \underbrace{H_k^* R_k^{-1} H_k}_{\text{Info. added by the meas.}}
 \end{aligned}$$

### Time update of $P_k^{-1}$

$$\begin{aligned}
 P_{k+1|k} &= F_k P_{k|k} F_k^* + G_k Q_k G_k^* \\
 P_{k+1|k}^{-1} &= (F_k P_{k|k} F_k^* + G_k Q_k G_k^*)^{-1} = \{A_k = F_k^{-*} P_{k|k}^{-1} F_k^{-1}\} \\
 &= (A_k^{-1} + G_k Q_k G_k^*)^{-1} \\
 &= A_k - \underbrace{A_k G_k (G_k^* A_k G_k + Q_k^{-1})^{-1} G_k^* A_k}_{\text{reduction in information when predicting}} \\
 &= (I - A_k G_k (G_k^* A_k G_k + Q_k^{-1})^{-1} G_k^*) A_k \\
 &= (I - B_k G_k^*) A_k,
 \end{aligned}$$

where  $B_k = A_k G_k (G_k^* A_k G_k + Q_k^{-1})^{-1}$ .

### State Recursions

**Definition:**  $\hat{a}_{k|k-1} = P_{k|k-1}^{-1} \hat{x}_{k|k-1}$  and  $\hat{a}_{k|k} = P_{k|k}^{-1} \hat{x}_{k|k}$

#### Measurement update:

$$\begin{aligned}
 \hat{x}_{k|k} &= \hat{x}_{k|k-1} + P_{k|k} H_k^* R_k^{-1} (y_k - H_k \hat{x}_{k|k-1}) \\
 P_{k|k}^{-1} \hat{x}_{k|k} &= P_{k|k}^{-1} \hat{x}_{k|k-1} + H_k^* R_k^{-1} y_k - H_k^* R_k^{-1} H_k \hat{x}_{k|k-1} \\
 &= P_{k|k-1}^{-1} \hat{x}_{k|k-1} + H_k^* R_k^{-1} y_k \\
 \hat{a}_{k|k} &= \hat{a}_{k|k-1} + H_k^* R_k^{-1} y_k
 \end{aligned}$$

#### Time update:

$$\begin{aligned}
 \hat{x}_{k+1|k} &= F_k \hat{x}_{k|k} \\
 \hat{a}_{k+1|k} &= P_{k+1|k}^{-1} \hat{x}_{k+1|k} = P_{k+1|k}^{-1} F_k \hat{x}_{k|k} = (I - B_k G_k^*) A_k F_k \hat{x}_{k|k} \\
 &= (I - B_k G_k^*) F_k^{-*} \hat{a}_{k|k}
 \end{aligned}$$

**Summary:**

**Measurement update:**

$$\hat{a}_{k|k} = \hat{a}_{k|k-1} + H_k R_k^{-1} y_k$$
$$P_{k|k}^{-1} = P_{k|k-1}^{-1} + H_k^* R_k^{-1} H_k$$

**Time update:**

$$\hat{a}_{k+1|k} = (I - B_k G_k^*) F_k^{-*} \hat{a}_{k|k}$$
$$P_{k+1|k} = (I - B_k G_k^*) A_k$$

where

$$A_k = F_k^{-*} P_{k|k} F_k^{-1}$$
$$B_k = A_k G_k (G_k^* A_k G_k + Q_k)^{-1}$$