Lecture #6

Gustaf Hendeby
gustaf.hendeby@liu.se

Version: 2023-12-06

Smoothing

In many applications an delayed estimate might be tolerable or off-line processing is interesting. This way, more data can be used in the estimate, providing

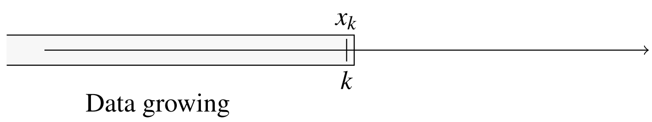
- higher accuracy
- more complex estimators.

Different Types of Smoothers

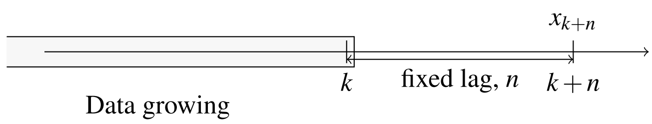
Assume:

$$\hat{x}_{l|k} = \text{llse of } x_l \text{ given } \{y_0, y_1, \dots, y_k\}$$

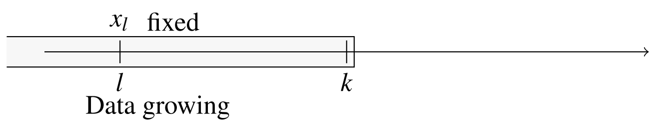
Filter:
$\hat{x}_{k|k}$



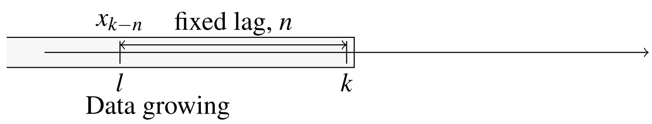
Predictor:
$\hat{x}_{k+n|k}$, $n > 0$



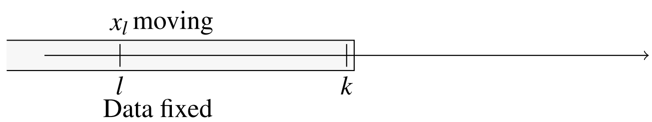
Fixed point smoother:
$\hat{x}_{l|k}$, l-fixed, k-growing



Fixed lag smoother:
$\hat{x}_{k-n|k}$, $n > 0$, n-fixed, k-growing



Fixed interval smoother:
$\hat{x}_{l|k}$, $k = M$, $0 \le l \le M$



Applications

- Fixed point smoothing: Initial condition estimation.
- Fixed lag smoothing: Online processing with delay.
- Fixed interval smoothing: Offline processing.

Fixed Point Smoothing (see ex 4 in HW #3)

Goal: Find $\hat{x}_{j|k}$ for all $k > j$, j-fixed

Trick: Introduce the augmented state vector $\begin{pmatrix} x_k^* & (x_k^a)^* \end{pmatrix}^*$ and the augmented state-space model

$$\begin{pmatrix} x_{k+1} \\ x_{k+1}^a \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_k \\ x_k^a \end{pmatrix} + \begin{pmatrix} G_k \\ 0 \end{pmatrix} w_k$$
$$y_k = \begin{pmatrix} H_k & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_k^a \end{pmatrix} + v_k.$$

Note, the dynamic transition of the augmented part of the state $x_k^a \equiv x_j$ is constant.

Apply the Kalman filter recursion

Gain:
$$\begin{pmatrix} K_{p,k} \\ K_{p,k}^a \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^a)^* \\ P_{k|k-1}^a & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} H_k^* \\ 0 \end{pmatrix} (H_k P_{k|k-1} H_k^* + R_k)^{-1},$$

where $P_{k|k-1}^{aa} = P_{k|k}^{aa} = \mathsf{E}\left[(x_j - \hat{x}_{j|k})(x_j - \hat{x}_{j|k})^*\right]$, yielding

$$K_{p,k} = F_k P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}$$
$$K_{p,k}^a = F_k P_{k|k-1}^a H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}$$

Covariance:
$$\begin{pmatrix} P_{k+1|k} & (P_{k+1|k}^a)^* \\ P_{k+1|k}^a & P_{k+1|k}^{aa} \end{pmatrix} = \begin{pmatrix} F_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^a)^* \\ P_{k|k-1}^a & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} F_k^* & 0 \\ 0 & I \end{pmatrix}$$
$$- \begin{pmatrix} K_{p,k} \\ K_{p.k}^a \end{pmatrix} \begin{pmatrix} H_k & 0 \end{pmatrix} \begin{pmatrix} P_{k|k-1} & (P_{k|k-1}^a)^* \\ P_{k|k-1}^a & P_{k|k-1}^{aa} \end{pmatrix} \begin{pmatrix} F_k^* & 0 \\ 0 & I \end{pmatrix}$$
$$+ \begin{pmatrix} G_k \\ 0 \end{pmatrix} Q_k \begin{pmatrix} G_k^* & 0 \end{pmatrix}$$

Remark: $K(HPH^* + R)K^* = K(HPH^* + R)(HPH^* + R)^{-1} HPF^* = KHPF^*$

$$P_{k+1|k} = F_k P_{k|k-1} F_k^* - K_{p,k} H_k P_{k|k-1} F_k^* + G_k Q G_k^* \quad \text{standard update}$$
$$P_{k+1|k}^a = P_{k|k-1}^a F_k^* - K_{p,k}^a H_k P_{k|k-1} F_k^*$$
$$P_{k+1|k}^{aa} = P_{k|k-1}^a - K_{p,k}^a H_k P_{k|k-1}^a$$

Estimate:

$$\hat{x}_{j|k} = \hat{x}_{j|k-1} + K_{p,k} e_k$$
$$e_k = y_k - \begin{pmatrix} H_k & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_{k|k-1} \\ \hat{x}^a_{k|k-1} \end{pmatrix} = y_k - H\hat{x}_{k|k-1}$$

Summary: Standard Kalman filter recursions

$$\begin{aligned}
\hat{x}_{j|k} &= \hat{x}_{j|k-1} + K^a_{p,k}(y_k - H_k\hat{x}_{k|k-1}) \\
K^a_{p,k} &= P^a_{k|k-1} H_k^* R_{e,k}^{-1} \\
P^a_{k+1|k} &= P^a_{k|k-1} F_k^* - K^a_{p,k} H_k P_{k|k-1} F_k^* \\
P^{aa}_{k+1|k} &= P^{aa}_{k|k-1} - K_{p,k} H_k P^a_{k|k-1}
\end{aligned}$$

Initial condition: $\hat{x}^a_{j|j-1} = \hat{x}_{j|j-1}$ and $P^a_{j|j-1} = P_{j|j-1}$

Remark:

- Most improvements are obtained after 2–3 time constants of the system.
- Time varying even for time invariant systems.

Fixed Lag Smoother

Goal: Find $\hat{x}_{k-n|k}$ and $P_{k-n|k}$, n-fixed, k-growing

Trick: Introduce the state-space model

$$\begin{pmatrix} x_{k+1} \\ x^{(1)}_{k+1} \\ x^{(2)}_{k+1} \\ \vdots \\ x^{(n+1)}_{k+1} \end{pmatrix} = \begin{pmatrix} F_k & 0 & \dots & & 0 \\ I & 0 & \ddots & & \\ 0 & I & & & \\ \vdots & \ddots & \ddots & & \\ 0 & \dots & & I & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x^{(1)}_k \\ x^{(2)}_k \\ \vdots \\ x^{(n+1)}_k \end{pmatrix} + \begin{pmatrix} G_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} w_k$$

$$y_k = \begin{pmatrix} H_k & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x^{(1)}_k \\ x^{(2)}_k \\ \vdots \\ x^{(n+1)}_k \end{pmatrix} + v_k$$

Note that: $x^{(1)}_{k+1} = x_k, x^{(2)}_{k+1} = x_{k-1}, \dots, x^{(n+1)}_{k+1} = x_{k-n}$, yielding $\hat{x}^{(i)}_{k+1|k} = x_{k-i+1|k}$, $i = 1, \dots, n+1$.

Hence: The one-step predictor of the augmented state-space model will give the l.l.s.e. of x_{k-i} given $\{y_j\}_{j=0}^k$ for $i = 0, \dots, n$.

Fixed Lag Smoothing Equations:

Standard Kalman filter recursions for the original state-space model.

1: $\hat{x}_{0|-1} = \hat{x}_{-i|-1} = 0$
2: $P^{(0)}_{k|k-1} = P_{k|k-1}$
3: **for** i=1,..., N **do**
4: $\quad \hat{x}^{(i+1)}_{k+1|k} = \hat{x}^{(i)}_{k|k-1} + K^{(i+1)}_{p,k} e_k$
5: $\quad K^{(i+1)}_{p,k} = P^{(i)}_{k|k-1} H_k^* R_{e,k}^{-1}$
6: $\quad P^{(i+1)}_{k+1|k} = P^{(i)}_{k|k-1} (F_k - K_{p,k} H_k)^*$
7: $\quad P_{k-i|k} = P_{k-i|k-1} - P^{(i)}_{k|k-1} H_k^* (K^{(i)}_{p,k})^*$
8: **end for**

Remark:

- Time invariant for time invariant state-space model.
- If n corresponds to 2–3 time constants almost all info has been included in the estimate $\hat{x}_{k-n|k}$, hence a fixed interval smoother gives marginal improvement.

Fixed Interval Smoother

See Bryson-Frazier and Rausch-Tung-Stribel formulas in the book.

Information Filter

If the initial state is completely unknown, then $P_0 \to \infty$, which may cause numerical problems. However, $P_0^{-1} = 0$ is quite natural. Propagating P_k^{-1} instead of P_k is referred to the information formulation of the Kalman filter.

Note: for Gaussian linear state-space systems, then P_k^{-1} is equivalent to the Fisher information matrix.

Derivation of the Information Filter

Assume:

- $S_k = 0$ (for simplicity)
- F_k non-singular (for simplicity)

Recall:

- $(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)DA^{-1}$
- Kalman filter gain: $K_{f,k} = P_{k|k-1} H_k^* R_{e,k}^{-1} = P_{k|k} H_k^* R_k^{-1}$

Measurement update of P_k^{-1}

$$
\begin{aligned}
P_{k|k} &= P_{k|k-1} - P_{k|k-1}H_k^*(H_kP_{k|k}H_k^* + R_k)^{-1}H_kP_{k|k-1} \\
&= (P_{k|k-1} + H_k^*R_k^{-1}H_k)^{-1} \\
P_{k|k}^{-1} &= P_{k|k-1}^{-1} + \underbrace{H_k^*R_k^{-1}H_k}_{\text{Info. added by the meas.}}
\end{aligned}
$$

Time update of P_k^{-1}

$$
\begin{aligned}
P_{k+1|k} &= F_kP_{k|k}F_k^* + G_kQ_kG_k^* \\
P_{k+1|k}^{-1} &= (F_kP_{k|k}F_k^* + G_kQ_kG_k^*)^{-1} = \{A_k = F_k^{-*}P_{k|k}^{-1}F_k^{-1}\} \\
&= (A_k^{-1} + G_kQ_kG_k^*)^{-1} \\
&= A_k - \underbrace{A_kG_k(G_k^*A_kG_k + Q_k^{-1})^{-1}G_k^*A_k}_{\text{reduction in information when predicting}} \\
&= (I - A_kG_k(G_k^*A_kG_k + Q_k^{-1})^{-1}G_k^*)A_k \\
&= (I - B_kG_k^*)A_k,
\end{aligned}
$$

where $B_k = A_kG_k(G_k^*A_kG_k + Q_k^{-1})^{-1}$.

State Recursions

Definition: $\hat{a}_{k|k-1} = P_{k|k-1}^{-1}\hat{x}_{k|k-1}$ and $\hat{a}_{k|k} = P_{k|k}^{-1}\hat{x}_{k|k}$

Measurement update:

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + P_{k|k}H_k^*R_k^{-1}(y_k - H_k\hat{x}_{k|k-1}) \\
P_{k|k}^{-1}\hat{x}_{k|k} &= P_{k|k}^{-1}\hat{x}_{k|k-1} + H_k^*R_k^{-1}y_k - H_k^*R_k^{-1}H_k\hat{x}_{k|k-1} \\
&= P_{k|k-1}^{-1}\hat{x}_{k|k-1} + H_k^*R_k^{-1}y_k \\
\hat{a}_k &= \hat{a}_{k|k-1} + H_k^*R_k^{-1}y_k
\end{aligned}
$$

Time update:

$$
\begin{aligned}
\hat{x}_{k+1|k} &= F_k\hat{x}_{k|k} \\
\hat{a}_{k+1|k} &= P_{k+1|k}^{-1}\hat{x}_{k+1|k} = P_{k+1|k}^{-1}F_k\hat{x}_{k|k} = (I - B_kG_k^*)A_kF_k\hat{x}_{k|k} \\
&= (I - B_kG_k^*)F_k^{-*}\hat{a}_{k|k}
\end{aligned}
$$

Summary:

Measurement update:

$$
\begin{aligned}
\hat{a}_{k|k} &= \hat{a}_{k|k-1} + H_k R_k^{-1} y_k \\
P_{k|k}^{-1} &= P_{k|k-1}^{-1} + H_k^* R_k^{-1} H_k
\end{aligned}
$$

Time update:

$$
\begin{aligned}
\hat{a}_{k+1|k} &= (I - B_k G_k^*) F_k^{-*} \hat{a}_{k|k} \\
P_{k+1|k} &= (I - B_k G_k^*) A_k
\end{aligned}
$$

where

$$
\begin{aligned}
A_k &= F_k^{-*} P_{k|k} F_k^{-1} \\
B_k &= A_k G_k (G_k^* A_k G_k + Q_k)^{-1}
\end{aligned}
$$